The LDU-Decomposition for the Fundamental Matrix of Time-Varying Systems

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Abstract—The variational equations of nonlinear dynamic systems are linear time-varying (LTV) by nature. In this article, we derive the LDU-decomposition for the fundamental matrix of these LTV systems. To that aim, the system matrix is first triangularized by successive Riccati transforms. Then, the diagonal matrix is substracted, followed by a procedure of repeated integration for the elements of the upper triangular matrix.

1. Introduction

As is well known, the variational equations of Poincaré play an important role in the stability theory of nonlinear dynamic systems [1]. To that aim, we focus on the following homogeneous linear time-varying (LTV) equation [2]

$$\dot{\boldsymbol{x}} = \mathbf{A}(t)\boldsymbol{x} \ . \tag{1}$$

Here, $\boldsymbol{x} = \boldsymbol{x}(t)$ denotes a sufficient small perturbation of the original nonlinear solution, while the $n \times n$ system matrix $\mathbf{A}(t)$ is the time-dependent Jacobian-matrix evaluated along a (possible chaotic) solution trajectory of the nonlinear dynamic system under study.

Until now, a general and constructive theory for the solution of LTV equations is not available. See for example [3], [4] and [5] for a historical context. However, since then several attempts have been made. Here, we mention the work presented in [6], [7], [8], [9], [10] and [11]. On its turn, this work was subsequently generalized by Van der Kloet and Neerhoff (see for example [12], [13], [14], [15] and [16]).

In this contribution, we present a new LDU-decomposition for the fundamental matrix solution of the LTV-equation (1).

In the next section, the lower triangular matrix $\mathbf{L}$ is obtained by successive Riccati transformations. In each step, a generalized characteristic equation has to be solved for the so-called dynamic eigenvalues. Apart from a slight different notation, the triangularization procedure is essentially the same as described earlier in for example [13] and [14]. However, from now on we proceed in a different way. Therefore, in Section 3, we first seperate the diagonal matrix $\mathbf{D}$. It is obtained in terms of exponential functions with the dynamic eigenvalues as arguments. Then, in Section 4, the elements of the upper triangular matrix $\mathbf{U}$ are obtained by repeated integration. Finally, in Section 5, the desired fundamental matrix $\boldsymbol{\Phi}$ is written as a product of $\mathbf{L}$, $\mathbf{D}$ and $\mathbf{U}$.

2. The Triangularization of the System Matrix

In this section we start with the LTV differential system (1). First, it is shown that this system can be transformed into

$$\dot{\boldsymbol{y}}(t) = \mathbf{B}(t)\boldsymbol{y}(t) \tag{2}$$

with $\mathbf{B}(t)$ an $n \times n$ upper triangle matrix. This is effectuated in $n-1$ successive steps. The first step transforms $\mathbf{A}(t)$ into a matrix whose elements in the n-th row are forced to zero, except the diagonal element. Then, the elements to the left of the diagonal in the last two rows are forced to zero. Continuing this procedure, yields the upper triangle matrix $\mathbf{B}$ in $n-1$ steps. Before the procedure is outlined, the following *Lemma* is proved.

Lemma:
Let the matrices $\mathbf{C}(t) = \{c_{ij}(t)\}$ and $\mathbf{X}(t) = \{x_{ij}(t)\}$ be partitioned as

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_{11} & \boldsymbol{c}_{12} \\ \boldsymbol{c}_{21}^T & c_{22} \end{bmatrix} \quad , \quad \mathbf{X} = \begin{bmatrix} \mathbf{I} & \boldsymbol{0} \\ \boldsymbol{x}^T & 1 \end{bmatrix} \tag{3}$$

with $\boldsymbol{c}_{12}$, $\boldsymbol{0}$ column vectors and $\boldsymbol{c}_{21}^T$, $\boldsymbol{x}^T$ row vectors.
Then it follows

$$\mathbf{X}^{-1}\mathbf{C}\mathbf{X} - \mathbf{X}^{-1}\dot{\mathbf{X}} = \begin{bmatrix} \mathbf{C}_{11} + \boldsymbol{c}_{12}\boldsymbol{x}^T & \boldsymbol{c}_{12} \\ \boldsymbol{0}^T & \lambda_n \end{bmatrix} \tag{4}$$

if and only if $\boldsymbol{x}^T$ satisfies

$$\begin{bmatrix} \boldsymbol{x}^T & -1 \end{bmatrix}^{\bullet} = \begin{bmatrix} \boldsymbol{x}^T & -1 \end{bmatrix} \begin{bmatrix} \lambda_n \mathbf{I} - \mathbf{C}_{11} & -\boldsymbol{c}_{12} \\ -\boldsymbol{c}_{21}^T & \lambda_n - c_{22} \end{bmatrix} . \tag{5}$$

Here, the matrix $\mathbf{X}$ is called a Riccati transform [17], while the matrices $\mathbf{C}$ and the right-hand side upper triangular matrix in (4) are called dynamic similar [12].
Proof:
Straightforward calculation shows

$$\mathbf{X}^{-1}\mathbf{C}\mathbf{X} - \mathbf{X}^{-1}\dot{\mathbf{X}} = \begin{bmatrix} \mathbf{C}_{11} + c_{12}x^T & c_{12} \\ -x^T\mathbf{C}_{11} - x^T c_{12}x^T + c_{22}x^T + c_{21}^T - \dot{x}^T & -x^T c_{12} + c_{22} \end{bmatrix}. \quad (6)$$

The right-hand side of (4) and (6) are equal, provided that $x^T = x^T(t)$ satisfies the following vector Riccati differential equation

$$\dot{x}^T = -x^T\mathbf{C}_{11} - x^T c_{12}x^T + c_{22}x_{21}^T + c_{21}^T , \quad (7)$$

while $\lambda_n = \lambda_n(t)$ follows as

$$\lambda_n = -x^T c_{12} + c_{22} . \quad (8)$$

Next, the constraints (7) and (8) are written as

$$\left.\begin{aligned} \dot{x}^T &= x^T(\lambda_n\mathbf{I} - \mathbf{C}_{11}) + c_{21}^T \\ 0 &= x^T c_{12} + \lambda_n - c_{22} \end{aligned}\right\} , \quad (9)$$

or, in matrix form

$$\begin{bmatrix} \dot{x}^T & 0 \end{bmatrix} = \begin{bmatrix} x^T & -1 \end{bmatrix} \begin{bmatrix} \lambda_n\mathbf{I} - \mathbf{C}_{11} & -c_{12} \\ -c_{21}^T & \lambda_n - c_{22} \end{bmatrix} . \quad (10)$$

Since $\begin{bmatrix} x^T & -1 \end{bmatrix}^\bullet = \begin{bmatrix} \dot{x}^T & 0 \end{bmatrix}$, equations (10) and (5) are equivalent.

□

To derive the $n-1$ successive steps that transform (1) into (2), we first write (1) as

$$\dot{y}_{n+1}(t) = \mathbf{B}_{n+1}(t)y_{n+1}(t) \quad (11)$$

with

$$y_{n+1}(t) = x(t) \quad \text{and} \quad \mathbf{B}_{n+1}(t) = \mathbf{A}(t) . \quad (12)$$

Now, the elementary Riccati transformations $\mathbf{L}_i$ are introduced with

$$y_{n+1-i}(t) = \mathbf{L}_{n-i}(t)y_{n-i}(t) \quad (i = 0, 1, 2, \ldots, n-2) , \quad (13)$$

such that $y_{n-i}(t)$ satisfies

$$\dot{y}_{n-i}(t) = \mathbf{B}_{n-i}(t)y_{n-i}(t) \quad (i = -1, 0, 1, \ldots, n-2) \quad (14)$$

with

$$\mathbf{B}_{n-i}(t) = \mathbf{L}_{n-i}^{-1}\mathbf{B}_{n-i+1}\mathbf{L}_{n-i} - \mathbf{L}_{n-i}^{-1}\dot{\mathbf{L}}_{n-i} . \quad (15)$$

Note that for $i = -1$ equation (14) equals (11). The remaining equations in (14) then follow simply by induction.
Next, the matrices $\mathbf{B}_{n-i+1}$ and $\mathbf{L}_{n-i}$ are partitionized as

$$\mathbf{B}_{n-i+1} = \begin{bmatrix} \mathbf{B}_{11}^{(n-i)\times(n-i)} & \mathbf{B}_{12}^{(n-i)\times i} \\ \mathbf{O} & \mathbf{B}_{22}^{i\times i} \end{bmatrix} , \ \mathbf{L}_{n-i} = \begin{bmatrix} \mathbf{P}_{n-i} & \mathbf{O} \\ \mathbf{O} & \mathbf{I} \end{bmatrix} \quad (16)$$

where $\mathbf{B}_{22}^{i\times i}$ is upper triangular. For $i = 0$ the matrix $\mathbf{B}_{22}^{i\times i}$ is empty and we have $\mathbf{B}_{n+1} = \mathbf{B}_{11}^{n\times n}$. Then, with respect to (15) and (16) we obtain

$$\mathbf{B}_{n-i} = \begin{bmatrix} \mathbf{P}_{n-i}^{-1}\mathbf{B}_{11}^{(n-i)\times(n-i)}\mathbf{P}_{n-i} - \mathbf{P}_{n-i}^{-1}\dot{\mathbf{P}}_{n-i} & \mathbf{P}_{n-i}^{-1}\mathbf{B}_{12}^{(n-i)\times i} \\ \mathbf{O} & \mathbf{B}_{22}^{i\times i} \end{bmatrix} . \quad (17)$$

Thus, if the last row of the matrix $[\mathbf{P}_{n-i}^{-1}\mathbf{B}_{11}^{(n-i)\times(n-i)}\mathbf{P}_{n-i} - \mathbf{P}_{n-i}^{-1}\dot{\mathbf{P}}_{n-i}]$ with the exception of the diagonal element becomes zero, then $\mathbf{B}_{n-i}$ has the same structure as $\mathbf{B}_{n-i+1}$ with an additional row of zeros. Now we choose

$$\mathbf{P}_{n-i} = \begin{bmatrix} \mathbf{I} & 0 \\ p_{n-i}^T & 1 \end{bmatrix} , \quad (18)$$

then the lemma can be applied with $\mathbf{P}_{n-i}$ for $\mathbf{X}$ and $\mathbf{B}_{11}^{(n-i)\times(n-i)}$ for $\mathbf{C}$, so that p_{n-i}^T and λ_{n-i} follow from

$$\begin{bmatrix} p_{n-i} & -1 \end{bmatrix}^\bullet = \begin{bmatrix} p_{n-i} & -1 \end{bmatrix} (\lambda_{n-i}\mathbf{I} - \mathbf{B}_{11}^{(n-i)\times(n-i)}) , \quad (19)$$

in which $i = 0, 1, \ldots, n-2$. At this stage, it is remarked that equation (19) can be interpreted as the generalization of the characteristic equation for LTI-systems ([18]). To show this shortly, consider again (7), (8) and (10), but now for a constant matrix $\mathbf{C}$ (c_{ij} is are constants). Then, equation (7) has a constant solution x^T that generates via (8) a constant λ_n. Also, if x^T is a constant, then the left-hand side of (10) equals 0^T. Next, it is noted that $\begin{bmatrix} x^T & -1 \end{bmatrix} \neq 0$, even if $x^T = 0^T$. As a consequence, the constant λ_n satisfies

$$\text{Det}(\lambda_n\mathbf{I} - \mathbf{C}) = 0 . \quad (20)$$

Hence λ_n equals an eigenvalue with respect of the matrix $\mathbf{C}$. Therefore, formula (19) denotes a set of $(n-1)$ generalized characteristic equations with respect to the LTV-system (11). The solution yields $(n-1)$ sets of vectors $p_{n-i}^T = p_{n-i}^T(t)$ and scalar functions $\lambda_{n-i} = \lambda_{n-i}(t)$ $(i = 0, 1, \ldots, n-2)$, respectively. The functions $\lambda_i = \lambda_i(t)$ are therefore generalizations of the constants λ_i, and are called dynamic eigenvalues, see [13] and [14]. Note, moreover, that λ_1 follows directly with

$$\lambda_1 = \text{Trace}\{\mathbf{A}(t)\} - \sum_{i=0}^{n-2} \lambda_{n-i} . \quad (21)$$

One can prove that the same structure holds for LTI-systems, see [19]. There, it is shown that the roots of $\text{Det}(\lambda_i\mathbf{I} - \mathbf{B}_{11}^{(n-i)\times(n-i)}) = 0$ are also roots of $\text{Det}(\lambda_n\mathbf{I} - \mathbf{B}_{n+1}) = 0$.
To summerize, there exists a Riccati transformation

$$y_{n+1} = \mathbf{L}y_2 \quad (22)$$

with

$$\mathbf{L} = \mathbf{L}_n \ldots \mathbf{L}_2 \quad (23)$$

such that (11) is transformed to

$$\dot{y}_2 = \mathbf{B}_2(t)y_2 \quad (24)$$

with $\mathbf{B}_2(t) = \{b_{ij}(t)\}$ an upper triangular matrix in which the main diagonal elements b_{ii} are the dynamic eigenvalues $\lambda_i = \lambda_i(t)$ for $i = 1, 2, \ldots, n$. Moreover, note that $\mathbf{L}(t) = \{l_{ij}\}$ is a lower triangular matrix. Since every $\mathbf{L}_i$ is a lower triangular matrix with ones on its main diagonal, it follows that the main diagonal elements of $\mathbf{L}$ equals to 1. Finally,note that (2) and (24) represent dynamic similar systems.

3. The Diagonal Matrix

We now consider (2) with $\mathbf{B} = \mathbf{B}_2$ of (24), thus $b_{ii} = \lambda_i$ with $\lambda_i = \lambda_i(t)$. First, we apply a transformation

$$\boldsymbol{y} = \mathbf{D}(t)\boldsymbol{z} \tag{25}$$

in which $\mathbf{D}$ denotes the diagonal matrix

$$\mathbf{D}(t) = \mathrm{Diag}\{e^{\gamma_i(t)}\}\,, \tag{26}$$

with $\gamma_i = \gamma_i(t)$ is a yet to be specified scalar function. Then, it follows

$$\dot{\boldsymbol{z}} = [\mathbf{D}^{-1}\mathbf{B}\mathbf{D} - \mathbf{D}^{-1}\dot{\mathbf{D}}]\boldsymbol{z}\,. \tag{27}$$

Since

$$\dot{\mathbf{D}} = \mathrm{Diag}\{\dot{\gamma}_i(t)\}\mathbf{D}\,, \tag{28}$$

equation (27) can be written as

$$\dot{\boldsymbol{z}} = \mathbf{D}^{-1}[\mathbf{B} - \mathrm{Diag}\{\dot{\gamma}_i(t)\}]\mathbf{D}\boldsymbol{z}\,. \tag{29}$$

With the choice

$$\dot{\gamma}_i(t) = \lambda_i(t) \text{ thus } \gamma_i(t) = \gamma_i(t_0) + \int_{t_0}^{t} \lambda_i(\tau)d\tau\,, \tag{30}$$

in which t_0 denotes an arbitrarily initial time. Then the system matrix in (29) follows as

$$\mathbf{D}^{-1}[\mathbf{B} - \mathrm{Diag}\{\dot{\gamma}_i(t)\}]\mathbf{D} = \begin{bmatrix} 0 & & b_{ij}e^{-(\gamma_i-\gamma_j)} \\ & \ddots & \\ 0 & & 0 \end{bmatrix}, \tag{31}$$

with a diagonal of zeros. Note also that with respect to (29), (30) and (31), we obtain

$$\dot{z}_i = b_{i,i+1}e^{-(\gamma_i-\gamma_{i+1})}z_{i+1} + \cdots + b_{in}e^{-(\gamma_i-\gamma_n)}z_n\,, \tag{32}$$

in which $i = n, n-1, \ldots, 1$, while $b_{ii} = 0$, so that we can solve successively $z_n, z_{n-1}, \ldots, z_1$. This is the procedure, similar to that followed in [14]. As a consequence, the solution of (29) with the choice (30) will be a product of an upper triangular matrix with the vector $\boldsymbol{z}(t_0)$ of initial values.

4. The Upper Triangular Matrix

However, we now proceed in a reversed order. To that aim, we first apply the transformation

$$\boldsymbol{z} = \mathbf{U}(t)\boldsymbol{w} \tag{33}$$

where $\mathbf{U}(t) = \{u_{ij}(t)\}$ denotes an upper triangular matrix with main diagonal elements $u_{ii} = 1$. Then, equation (29) is transformed into

$$\dot{\boldsymbol{w}} = \mathbf{U}^{-1}\Big(\mathbf{D}^{-1}[\mathbf{B} - \mathrm{Diag}\{\dot{\gamma}_i(t)\}]\mathbf{D}\mathbf{U} - \dot{\mathbf{U}}\Big)\boldsymbol{w}\,. \tag{34}$$

In view of (31) we then obtain the upper triangular matrix

$$\mathbf{V}(t) = \mathbf{D}^{-1}[\mathbf{B} - \mathrm{Diag}\{\dot{\gamma}_i(t)\}]\mathbf{D}\mathbf{U} - \dot{\mathbf{U}} \tag{35}$$

with the main diagonal elements $v_{ii} = 0$ and

$$v_{ij} = \sum_{k=i+1}^{j-1} b_{ik}e^{-(\gamma_i-\gamma_k)}u_{kj} + b_{ij}e^{-(\gamma_i-\gamma_j)} - \dot{u}_{ij}\ (j \geq i+1)\,. \tag{36}$$

In view of (34), the requirement $v_{ij} = 0$ $(j \geq i+1)$ induces $\dot{\boldsymbol{w}}(t) = \boldsymbol{0}$, and hence $\boldsymbol{w}(t) = \boldsymbol{w}(t_0)$. Moreover, the following set of expressions for $\dot{u}_{ij}$ is obtained

$$\left.\begin{aligned} \dot{u}_{i,i+1} &= b_{i,i+1}e^{-(\gamma_i-\gamma_{i+1})} \\ \dot{u}_{i,i+2} &= b_{i,i+2}e^{-(\gamma_i-\gamma_{i+2})} + b_{i,i+1}e^{-(\gamma_i-\gamma_{i+1})}u_{i+1,i+2} \\ &\vdots \\ \dot{u}_{i,n} &= b_{i,n}e^{-(\gamma_i-\gamma_n)} + b_{i,n-1}e^{-(\gamma_i-\gamma_{n-1})}u_{n-1,n} + \\ &\quad + \cdots + b_{i,i+1}e^{-(\gamma_i-\gamma_{i+1})}u_{i+1,n} \end{aligned}\right\}. \tag{37}$$

in which $i = 1, 2, \ldots, n-1$. This set allows analytic solutions by repeated integration. For example

$$\left.\begin{aligned} u_{i,i+1} &= u_{i,i+1}(t_0) + \int_{t_0}^{t} b_{i,i+1}(\tau)e^{-[\gamma_i(\tau)-\gamma_{i+1}(\tau)]}d\tau \\ u_{i,i+2} &= u_{i,i+2}(t_0) + \int_{t_0}^{t} b_{i,i+2}(\tau)e^{-[\gamma_i(\tau)-\gamma_{i+2}(\tau)]}d\tau + \\ &\quad + \int_{t_0}^{t} b_{i,i+1}(\tau)e^{-[\gamma_i(\tau)-\gamma_{i+2}(\tau)]}\times \\ &\quad \times \int_{t_0}^{\tau} b_{i+1,i+2}(\tau_1)e^{-[\gamma_i+1(\tau_1)-\gamma_{i+2}(\tau_1)]}d\tau_1 d\tau \end{aligned}\right\}, \tag{38}$$

in which t_0 again denotes the initial time.

5. The Fundamental Matrix

Finally, the solution for the LTV system (1) can be written as

$$\boldsymbol{x}(t) = \mathbf{L}(t)\mathbf{D}(t)\mathbf{U}(t)\boldsymbol{w}(t_0)\,, \tag{39}$$

where we have used $\boldsymbol{w}(t) = \boldsymbol{w}(t_0)$, or, in view of the transformations (22), (25), (33) and the type of the slutions (38), in terms of the desired fundamental matrix $\boldsymbol{\Phi} = \boldsymbol{\Phi}(t)$ as

$$\boldsymbol{x}(t) = \boldsymbol{\Phi}(t)\boldsymbol{x}(0) \tag{40}$$

with

$$\mathbf{\Phi}(t) = \mathbf{L}(t)\mathbf{D}(t)\mathbf{U}(t)\,, \tag{41}$$

or, with the arbitrary initial time t_0, as

$$\boldsymbol{x}(t) = \mathbf{\Phi}(t, t_0)\boldsymbol{x}(t_0) = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(t_0)\boldsymbol{x}(t_0)\,, \tag{42}$$

in which

$$\mathbf{\Phi}^{-1}(t) = \mathbf{U}^{-1}(t)\mathbf{D}^{-1}(t)\mathbf{L}^{-1}(t)\,. \tag{43}$$

As a simple illustration, consider the time-invariant equation (compare equation (11) in [16])

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \boldsymbol{x} \quad \rightleftarrows \quad \dot{\boldsymbol{x}} = \mathbf{A}\boldsymbol{x}\,. \tag{44}$$

With respect to (12) and (13) the transformation

$$\boldsymbol{x} = \begin{bmatrix} 1 & 0 \\ p & 1 \end{bmatrix} \boldsymbol{y} \quad \rightleftarrows \quad \boldsymbol{x} = \mathbf{L}\boldsymbol{y} \tag{45}$$

will give, as described in lemma 1, with $p = -2$

$$\dot{\boldsymbol{y}} = \begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix} \boldsymbol{y} \quad \rightleftarrows \quad \dot{\boldsymbol{y}} = \mathbf{B}\boldsymbol{y}\,. \tag{46}$$

The transformations (25) and (33) are applied in the next steps and will yield according to (41)

$$\mathbf{\Phi}(t) = \mathbf{LDU} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} 1 & u(t) \\ 0 & 1 \end{bmatrix}, \tag{47}$$

where $u(t)$ is obtained with (38) as $u(t) = -e^{-t}$.

6. Conclusion

This article presents a new procedure for the construction of the LDU-decomposition for the fundamental matrix of LTV-systems. First, the **L**-matrix is obtained by repeated Riccati transforms. Then, the **D**-matrix is obtained as exponential functions with dynamic eigenvalues as arguments. Finally, the elements of **U** follow by repeated integration.

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