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SOME ANALYTIC CALCULATIONS OF THE CHARACTERISTIC EXPONENTS

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As is well known, the variational equations of nonlinear dynamic systems are linear time-varying (LTV) by nature. In the modal solutions for these LTV equations, the earlier introduced dynamic eigenvalues play a key role. They are closely related to the Lyapunov- and Floquet-exponents of the corresponding nonlinear systems.

In this contribution, we present some simple examples for which analytic solutions exist. It is also demonstrated by example how the classical linear time-invariant (LTI) solutions are related to the equilibrium points of the general LTV solutions.

Keywords: LTV-systems; dynamic eigenvalues; Lyapunov- and Floquet-exponents.

1. Introduction

As is well known, the variational equations of Poincaré play an important role in the stability theory of nonlinear dynamic systems [Minorsky, 1987]. To that aim, we focus on the following homogeneous linear time-varying (LTV) equation [Neerhoff & van der Kloet, 2001b]

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}. \qquad (1)$$

Here, $\mathbf{x} = \mathbf{x}(t)$ denotes a sufficient small perturbation of the original nonlinear solution, while $\mathbf{A}(t)$ is the time-dependent Jacobian-matrix evaluated along a solution trajectory of the nonlinear dynamic system under study.

In a number of papers, van der Kloet and Neerhoff contributed to a general and constructive theory for the modal solutions of the LTV equation (1). See therefore [van der Kloet & Neerhoff, 2000, 2001, 2003, 2004] and [Neerhoff & van der Kloet, 2001a, 2001b]. In fact, their theory generalizes the work presented in [Wu, 1980; Kamen, 1988; Zhu & Johnson, 1991; Zhu, 1997].

In their approach the dynamic eigenvectors and dynamic eigenvalues are central concepts. The dynamic eigenvalues turn out to be the solutions of a generalized characteristic equation with the form of a Riccati differential equation. They are closely related to the Lyapunov- and Floquet-exponents, respectively.

In this paper, the theory is validated by two simple examples. The first example demonstrates how the classical LTI theory is contained in the LTV approach. Furthermore, for both examples, analytic solutions for the dynamic eigenvalues, dynamic eigenvectors, as well as for the characteristic exponents are obtained.

2. Mathematical Background

We are studying elementary solutions of Eq. (1) of the modal form

$$\mathbf{x}(t) = \mathbf{u}(t) \exp[\gamma(t)], \tag{2}$$

where $\mathbf{u} = \mathbf{u}(t)$ denotes a time-varying amplitude-vector, while the time-varying phase $\gamma = \gamma(t)$ is related to a time-varying frequency $\lambda = \lambda(t)$, given by

$$\lambda(t) = \dot{\gamma}(t) \quad \text{with } \gamma(t) = \int_0^t \lambda(\tau) d\tau. \tag{3}$$

Then, substitution of (2) into (1) yields the so-called *dynamic* eigenvalue problem [van der Kloet & Neerhoff, 2000]

$$[\mathbf{A}(t) - \lambda(t)\mathbf{I}]\mathbf{u}(t) = \dot{\mathbf{u}}(t), \tag{4}$$

where $\mathbf{I}$ denotes the identity matrix. In the context of expression (4), $\mathbf{u} = \mathbf{u}(t)$ is called a *dynamic* eigenvector and $\lambda = \lambda(t)$ the corresponding *dynamic* eigenvalue, respectively. In this respect it has to be understood that, although the classical eigenvalues of the time-varying matrix $\mathbf{A}(t)$ are time-dependent too, in general they do not have the direct physical interpretation as their dynamical counterparts.

For a second-order system Eq. (1) reads

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{5}$$

In order to obtain solutions for the dynamic eigenvalues of this second-order system, we apply the Lyapunov–Riccati transformation [Smith, 1987]

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ p(t) & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \tag{6}$$

that brings system (5) into the *dynamic* similar triangular system [van der Kloet & Neerhoff, 2000]

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \lambda_1(t) & a_{12}(t) \\ 0 & \lambda_2(t) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{7}$$

iff $p(t)$ satisfies the Riccati differential equation

$$\dot{p} = -a_{12}p^2 - (a_{11} - a_{22})p + a_{21} \tag{8}$$

and where $\lambda_1 = \lambda_1(t)$ and $\lambda_2 = \lambda_2(t)$, given by

$$\lambda_1 = a_{11} + a_{12}p \quad \text{and} \quad \lambda_2 = -a_{12}p + a_{22}, \tag{9}$$

denote the dynamic eigenvalues corresponding to the original LTV system (5).

3. Two Examples

The first example concerns the following second order LTI system equation

$$\ddot{x} + 5\dot{x} + 6x = 0. \tag{10}$$

With the substitution $x = x_1$ and $\dot{x} = x_2$ the state-space description of Eq. (10) is obtained as

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{11}$$

For this system, Eqs. (7)–(9) respectively become

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \lambda_1(t) & 1 \\ 0 & \lambda_2(t) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \tag{12}$$

$$-\dot{p} = p^2 + 5p + 6 \tag{13}$$

and

$$\lambda_1 = p \quad \text{and} \quad \lambda_2 = -p - 5. \tag{14}$$

Substitution of $\lambda = \lambda_{1,2}$ in the Riccati Eq. (13) yields the *generalized* characteristic equation for second-order LTV systems [Neerhoff & van der Kloet, 2001b]

$$-\dot{\lambda} = \lambda^2 + 5\lambda + 6. \tag{15}$$

In Fig. 1 the dynamic route of Eq. (13) is depicted. It is observed that the solution of (13) has two equilibrium points $p(\infty) = -2$ and $p(-\infty) = -3$. This also follows from the classical LTI theory. Then $\dot{p} = 0$ in (13). Furthermore, it is observed that

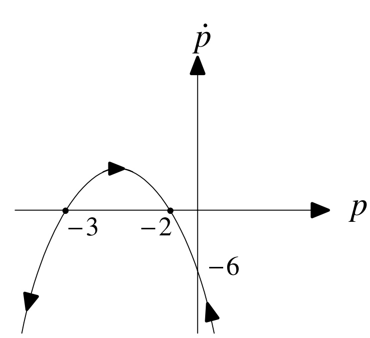

Fig. 1. The dynamic route of Eq. (13).

$p(-\infty)$ is an unstable equilibrium point, while $p(\infty)$ is a stable one.

With the substitution

$$p(t) = -2 + \xi(t) \tag{16}$$

into the Riccati Eq. (13) it follows that $\xi = \xi(t)$ satisfies the Bernoulli equation

$$-\dot{\xi} = \xi + \xi^2, \tag{17}$$

which is solved by the substitution $\xi = \eta^{-1}$. Then, it can be shown that the expression

$$\begin{aligned} p(t) = -2 + {}&[\exp\{-(t + \ln C)\}] \\ &\times [1 - \exp\{-(t + \ln C)\}]^{-1} \end{aligned} \tag{18}$$

satisfies the original Riccati Eq. (13). Here, the constant C is given by

$$C = [1 + \{2 + p(t_1)\}^{-1}]\exp(-t_1), \tag{19}$$

where t_1 denotes the initial time with respect to the Riccati differential Eq. (13). Figure 2 depicts the graph of $p = p(t)$. It is noted that, although any solution of (13) will tend to $p(\infty) = -2$, solutions with $p(t_1) < -3$ display a discontinuity, reflecting a finite escape time. Next, the dynamical eigenvalues follow from (14) as

$$\left.\begin{aligned} \lambda_1(t) = -2 + {}&[\exp\{-(t + \ln C)\}] \\ &\times [1 - \exp\{-(t + \ln C)\}]^{-1} \\ \lambda_2(t) = -3 - {}&[\exp\{-(t + \ln C)\}] \\ &\times [1 - \exp\{-(t + \ln C)\}]^{-1} \end{aligned}\right\}. \tag{20}$$

The *mean value* of the dynamic eigenvalues over an infinite period of time are directly related to the

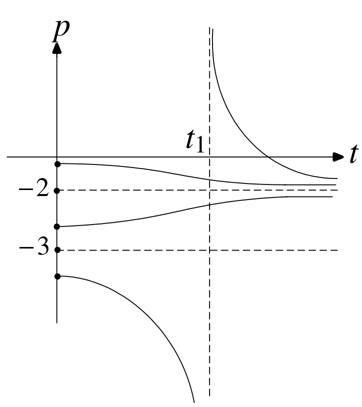

Fig. 2. Solutions of the Riccati equation for different initial values.

Lyapunov exponent χ_1 and χ_2 as [van der Kloet & Neerhoff, 2004]

$$\chi_{1,2} = \lim_{t\to\infty} \text{Re}\left[t^{-1}\int_0^{\text{t}} \lambda_{1,2}(\tau)\,d\tau\right]. \tag{21}$$

Now, it is easily shown that

$$\begin{aligned} \lim_{t\to\infty} t^{-1} \int_0^t &[\exp\{-(\tau + \ln C)\}] \\ &\times [1 - \exp\{-(\tau + \ln C)\}]^{-1} d\tau = 0. \end{aligned} \tag{22}$$

Thus, the Lyapunov exponents $\chi_{1,2}$ are obtained as

$$\chi_1 = \text{Re}\, L_1 = -2 \quad \text{and} \quad \chi_2 = \text{Re}\, L_2 = -3. \tag{23}$$

Note that, although the dynamic eigenvalues (20) depend on the initial time t_1 with respect to the Riccati Eq. (13), the *mean values* in (23) are *independent* of t_1.

As a next step, substitution of (20) in (12) yields $y_1 = y_1(t)$ and $y_2 = y_2(t)$, which on their turn yields in view of (6) the solutions $x_1 = x_1(t)$ and $x_2 = x_2(t)$. Then, after some elementary manipulations, we arrive at the well-known LTI solution given by (2) as

$$\left.\begin{aligned} x_1(t) &= u_1(0)\exp(-2t) + u_2(0)\exp(-3t) \\ x_2(t) &= -2u_1(0)\exp(-2t) - 3u_2(0)\exp(-3t) \end{aligned}\right\}, \tag{24}$$

in which $u_1(0) = 3x_1(0) + x_2(0)$ and $u_2(0) = -2x_1(0) - x_2(0)$. Finally, we obtain

$$\begin{aligned} x(t) = {}&\{3x(0) + \dot{x}(0)\}\exp(-2t) - \{2x(0) + \dot{x}(0)\} \\ &\times \exp(-3t). \end{aligned} \tag{25}$$

The second example concerns the following periodic system equation

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 + \alpha\cos^2(t) & 1 - \alpha\sin(t)\cos(t) \\ -1 - \alpha\sin(t)\cos(t) & -1 + \alpha\sin^2(t) \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{26}$$

in which α is some constant.

By applying the Lyapunov–Riccati transformation (6), Eq. (26) goes into

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \alpha - 1 - \tan(t) & 1 - \alpha\sin(t)\cos(t) \\ 0 & -1 + \tan(t) \end{bmatrix} \times \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{27}$$

iff $p = p(t)$ satisfies the Riccati differential Eq. (8), here obtained as

$$\dot{p} = -[1 - \alpha \sin(t)\cos(t)]p^2 - \alpha[\cos^2(t) - \sin^2(t)]p - [1 + \alpha \sin(t)\cos(t)]. \tag{28}$$

Next, it is easily verified that

$$p(t) = -\tan(t) \tag{29}$$

is a solution of (28). Then, after backward integration of the triangular system (27), followed by the application of the Lyapunov–Riccati transformation (6) combined with (29), we arrive at the following Floquet representation

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \times \begin{bmatrix} \exp[(\alpha - 1)t] & 0 \\ 0 & \exp(-t) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}. \tag{30}$$

Furthermore, the dynamic eigenvalues $\lambda_{1,2} = \lambda_{1,2}(t)$ follow from (27) as

$$\lambda_1(t) = \alpha - 1 - \tan(t) \quad \text{and} \quad \lambda_2(t) = -1 + \tan(t). \tag{31}$$

Finally, the Floquet exponents $\mu_{1,2}$ are related to the *mean values* of the dynamic eigenvalues $\lambda_{1,2}$ by [van der Kloet & Neerhoff, 2004]

$$\mu_{1,2} = \mathrm{Re}\left[T^{-1} \int_0^{\mathrm{T}} \lambda_{1,2}(\tau)\mathrm{d}\tau \right], \tag{32}$$

in which T denotes the system period $T = \pi$. Thus

$$\mu_1 = \alpha - 1 \quad \text{and} \quad \mu_2 = -1, \tag{33}$$

which is also confirmed by the exponentials in (30).

4. Conclusions

It is demonstrated by simple examples that constant (LTI) and periodic system solutions are generated as special cases of a general LTV modal theory. Although in principle, the dynamic eigenvalues are not unique, it is shown that, as expected, the charateristic eponents and the final solutions are.

As shown earlier, our modal theory requires solutions of Riccati equations. However, in nearly any practical application, analytic solutions are not available. Then, approximation methods have to be developed. In [van der Kloet & Neerhoff, 2000] such a method is described. In it, an iterative differentiation scheme is applied. Hence, the method is less suitable for numerical purposes. In this respect, see [van der Kloet *et al.*, 1999] for a more promising approach. It includes an algorithm based on repeated integrations.

Finally, approximated expressions can be obtained by asymptotic expansion [Smith, 1987; Eastham, 1989].

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