SOME REMARKS ON MODAL SOLUTIONS FOR SECOND AND THIRD ORDER TIME-VARYING SYSTEMS

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ABSTRACT

For linear time-varying systems are modal solutions obtained. This process needs a triangularization of the state-space matrix. This involves a set of Riccati differential equations, which generalize the characteristic equations for time-invariant systems. As examples third and second order systems are treated.

1. INTRODUCTION

Electronic circuits containing elements as transistors, operational amplifiers and so on are modelled mathematically as a set of nonlinear algebraic differential equations. These serve to determine a bias point (as in the case of an amplifier) or a bias trajectory (as for oscillators). Properties of the circuit as stability, distortion, noise and so on can be derived using a set of equations which express the variations on the original set of nonlinear equations. This set of equations expressing the variations is linear. For circuits with a fixed bias point the variational equations are also time-invariant, while for circuits with a bias trajectory the variational equations are time-varying. In the latter case the coefficients of the equations are periodic for oscillator circuits. For circuits with a chaotic behavior the dependency of the coefficients on time may be arbitrarily.

As a consequence modal solutions for the variational equations should not only be developped for linear time-invariant circuits and systems, but also for linear time-varying circuits and systems. In [1] such a modal solution for linear time-varying systems has been given. It is proved there that the modal solution can be written as a weighted sum of exponentials. Each weight is time-dependent and, moreover, the arguments of the exponentials are, in general, nonlinear functions of time. This general shape of a mode goes back to Van der Pol [2]. For time-invariant systems the weights reduce to constants and the arguments of the exponentials become linear functions of time. The constants of proportionality in these linear fuctions are recognized as eigenvalues (and their inverses are commonly known as time-constants).

Recent contributions to the theory for time-varying systems are given by Kamen [3], Zhu [4] and Wu [5]. Kamen introduced the concept of a right pole for a second order linear differential equation with time-varying coefficients. It is obtained by writing the second order system as a product of two first order systems, i.e. the Floquet decomposition. Kamen concludes that the integral of the right pole is the argument of an exponential function in the modal solution, while the weight is a constant. Furthermore he concluded that the right pole satisfies an equation of Riccati. Zhu generalized the work of Kamen for general order scalar linear differential equations. Someway, the conclusions of Kamen were justified and, moreover, he shew that a type of characteristic equations was justified. Solutions of characteristic equations are commonly known as eigenvalues and this gives a direct connection with the older work of Wu. That approach formulates first an eigenvalue problem from which the same type of eigenvalues are introduced as in the work of Zhu. It must be remarked that the work of Wu is based on a state-space description of the system and, thus, is more general than the approach of Zhu.

Zhu found that an n-th order system can be characterized by a set of $n-1$ characteristic equations. This furnishes $n-1$ right poles, and each of them yielding an argument for the exponential in a mode of the modal solution. The last right pole follows by algebraic operations. In [6] it is shown that this structure also holds for linear time-invariant systems (as should be the case for less general systems). For linear time-invariant systems there is one characteristic equation yielding all eigenvalues, while all the other characteristic equations yield only a subset of all eigenvalues.

The purpose of this paper is first to show shortly the basic theory for obtaining modal solutions of time-varying systems in terms of dynamic eigenvalues and dynamic eigenvectors as generalizations of the classical notion of eigenvalues and eigenvectors for time-invariant systems (section 2). Moreover it is demonstrated the manner how the differential equation of Riccati enters this theory, and, how (the solution of) the Riccati equations are related to (the solution of) the characteristic equations. This clearly shows that the solution of the Riccati equation involves the solution of the characteristic equation. The reverse does not hold. Moreover it is shown that only for the time-invariant case the solution of the characteristic can be found independent of the solution of the Riccati equation.

In section 3 this is demonstrated for a third order system. It is shown there that there are two coupled differential equations of Riccati. These reduce to two quadratic algebraic forms for time- invariant systems, which can have at most three points of intersection for the third order system. These points are the solution of the characteristic equation of the third order system. It is also demonstrated that for positive coefficients in the differential equations describing the system, the mentioned points of intersection are located in such a manner that the real eigenvalues are negative.

Also the Riccati equation for a second order system is exercised. In section 4 the concepts introduced in section 2 are exercised for a second order system. It is demonstrated that the Riccati differential equation has two points of equilibrium, one stable and the other unstable. These points are the solutions of the characteristic equation in the time-invariant case. By this approach the unstability in numerical processes for finding roots of a polynomial equation is explained. Moreover, the ordening of the roots of the characteristic equation is demonstrated.

It is also explained that the case of equal eigenvalues is not an exception, but fits in the general theory for time-varying systems. It is shown that for the Jordan matrix with the aid of a time-dependent transformation a diagonal matrix results.

As a consequence it may be concluded (section 5) that with the method followed a complete generalization for time-varying systems of the classical theory for time-invariant systems is obtained. It appears also that the mean values of the dynamic eigenvalues in the time-varying approach are equal to the classical eigenvalues in the time-invariant situation.

2. BASIC THEORY

Consider a system of linear time-varying differential equations

$$\dot{\boldsymbol{x}}(t) = \mathbf{A}(t)\boldsymbol{x}(t) \tag{1}$$

with $\boldsymbol{x} \in \mathbb{C}^n$, the set of all complex-valued vector functions, and $\mathbf{A} \in \mathbb{R}^{n\times n}$, the set of all real-valued matrix functions. It is assumed that that the elements $a_{ij}(t)$ are such that the solution of (1) is unique, given an initial value

$$\boldsymbol{x}(0) = \boldsymbol{x}_o \tag{2}$$

The system (1) can be transformed to a triangular system

$$\dot{\boldsymbol{y}}(t) = \mathbf{B}(t)\boldsymbol{y}(t) \tag{3}$$

where the elements $b_{ij}(t)$ of $\mathbf{B}(t)$ satisfies for $i > j$

$$b_{ij}(t) = 0 \tag{4}$$

Further we introduce the notation

$$b_{ii}(t) = \lambda_i(t) \tag{5}$$

As a consequence it can be shown that (3) has solutions

$$\boldsymbol{y}_i(t) = [\alpha_{1i}(t),\ldots,\alpha_{i-1,i}(t),1,0,\ldots,0]^T e^{\gamma_i(t)} \tag{6}$$

where

$$\gamma_i(t) = \int_0^t \lambda_i(\tau)d\tau \tag{7}$$

It is clear that these solutions are linearly independent. Moreover, they contain one exponential function. So the set $\{\boldsymbol{y}_1(t),\ldots,\boldsymbol{y}_n(t)\}$ constitutes a set of modal solutions.

If the procedure is followed for a system with constant coefficients, then it will appear that $\mathbf{A}$ and $\mathbf{B}$ are similar. Then, also, the functions λ_i reduce to constants, the eigenvalues of both $\mathbf{A}$ and $\mathbf{B}$.

We will call the finction $\lambda_i(t)$ dynamic eigenvalues, and the vector $[\alpha_{1i},\ldots,\alpha_{i-1,i},1,0,\ldots,0]^T$ in the corresponding modal solution a dynamic eigenvector with respect to $\mathbf{B}(t)$.

The transformation of (1) into (3) is performed by using

$$\boldsymbol{x}(t) = \mathbf{R}(t)\boldsymbol{y}(t) \tag{8}$$

with $\mathbf{R}(t)$ a lower triangular matrix. This matrix has to be constructed in a specific way, see section 3 and 4 for a third and a second order system. The general case proceeds along the same lines. It will also be clear that $\mathbf{R}(t)[\alpha_{1i},\ldots,\alpha_{i-1,i},1,0,\ldots,0]^T$ represents the dynamic eigenvectors with respect to $\mathbf{A}(t)$.

One can show that $\mathbf{R}(t)$ is the product

$$\mathbf{R}(t) = \mathbf{R}_n(t)\cdots\mathbf{R}_2(t) \tag{9}$$

with

$$\mathbf{R}_{n-i} = \begin{bmatrix} \mathbf{I}_{n-i-1} & \boldsymbol{0} & \mathbf{0} \\ \boldsymbol{p}_{n-i}^T & 1 & \boldsymbol{0}^T \\ \mathbf{0} & \boldsymbol{0} & \mathbf{I}_i \end{bmatrix} \tag{10}$$

where $\boldsymbol{p}_{n-i}^T$ has to satisfy a Riccati differential equation in order that $\mathbf{B}(t)$ becomes triangular.

3. THIRD-ORDER SYSTEM

In this section the equation

$$\dddot{x} + a_1(t)\ddot{x} + a_2(t)\dot{x} + a_3(t)x = 0 \tag{11}$$

will be discussed. First the state-space equation will be given as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^{\bullet} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3(t) & -a_2(t) & -a_1(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{12}$$

The read-out equation

$$x(t) = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{13}$$

gives the connection with (11).

The procedure outlined in section 2 gives first a transformation

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ p_{31}(t) & p_{32}(t) & 1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} \tag{14}$$

This yields

$$\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}^{\bullet} = \begin{bmatrix} 0 & 1 & 0 \\ p_{31}(t) & p_{32}(t) & 1 \\ 0 & 0 & \lambda_3(t) \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} \tag{15}$$

where $p_{31}(t)$, $p_{32}(t)$ and $\lambda_3(t)$ has to satisfy

$$\left.\begin{aligned} \dot{p}_{31} &= -(p_{32}+a_1)p_{31} - a_3 \\ \dot{p}_{32} &= -(p_{32}+a_1)p_{32} - p_{31} - a_2 \end{aligned}\right\} \tag{16}$$

$$\lambda_3 = -p_{32} - a_1 \tag{17}$$

The differential equations (16) are Riccati equations because the righthand sides have degree 2.

The third line in (15) learns that λ_3 is a generalization of the eigenvalue for a time-invariant system; it is called a dynamic eigenvalue. With the aid of (16) and (17) it is easy to derive that λ_3 is a solution of

$$\lambda_3^3 + a_1\lambda_3^2 + (a_2+\dot{p}_{32})\lambda_3 + (a_3+\dot{p}_{31}) = 0 \tag{18}$$

So λ_3 obeys a generalized characteristic equation. To solve (18) the equations (16) has to be solved. So the Riccati equations (16) appear to be of more fundamental meaning than

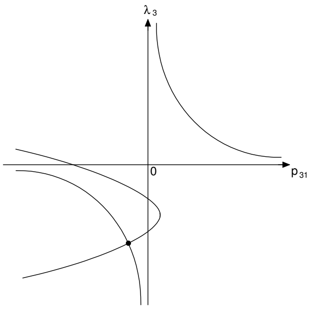

Figure 1: One point of intersection.

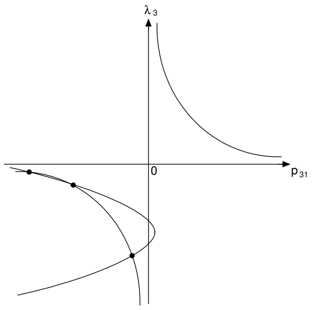

Figure 2: Three points of intersection.

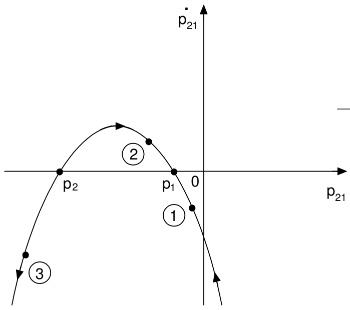

Figure 3: Dynamic route of (32).

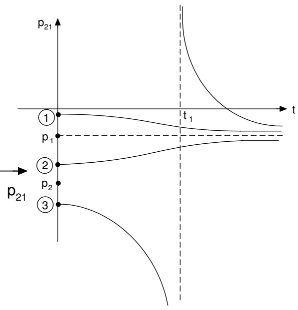

Figure 4: Behaviour of the solution of (32) for different initial values.

the characteristic equation. Moreover, results in system theory which are based on the characteristic equation are already included in (16).
As an example, we consider a linear time-invariant system in which all coefficients a_i are greater than zero. For such a system we have

$$\dot{p}_{32} = \dot{p}_{31} = 0 \tag{19}$$

So (18) reduces to the classical characteristic equation and (16) reduces to

$$\left.\begin{aligned} (p_{32}+a_1)p_{31} &= a_3 \\ p_{31} &= -(p_{32}+a_1)p_{32}-a_2 \end{aligned}\right\} \tag{20}$$

Using (17) as a change in the frame of reference, we can rewrite (20) as

$$\left.\begin{aligned} \lambda_3 p_{31} &= a_3 \\ p_{31} &= -\lambda_3^2 - a_1\lambda_3 - a_2 \end{aligned}\right\} \tag{21}$$

It is recognized that (21) gives a hyperbola and a parabola. They are sketched in the figures 1 and 2. Note that the parabola intersects the axis $p_{31} = 0$ in two points only if $a_2 < \frac{1}{4}a_1^2$. For both points of intersection we have $\lambda_3 < 0$. As a consequence, the parabola has no points in common with the first quadrant of a (p_{31}, λ_3)-plane. So all points of intersection of the hyperbola and the parabola in (21) are located in the third quadrant. They all fulfil the property $\lambda_3 < 0$ as might be expected for a system with $a_i > 0 (i = 1,2,3)$.
As a consequence we note that infomation contained in the characteristic equation (18) is already contained in the set of Riccati equations (16). The use of a transformation

$$\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ p_{21}(t) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \tag{22}$$

will transform (15) into

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}^\bullet = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \tag{23}$$

if

$$\dot{p}_{21} = -p_{21}^2 + p_{32}p_{21} + p_{31} \tag{24}$$

and $\lambda_1 = p_{21}$. It is evident that (22) can be split up into two independent sets. First we have a second order system, which is not homogeneous

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}^\bullet = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} y_3 \tag{25}$$

and secondly

$$\dot{y}_3 = \lambda_3 y_3 \tag{26}$$

As is known the characteristic equation is determined only by the homogeneous part.

4. SECOND ORDER SYSTEMS

In this section

$$\ddot{x} + a_1(t)\dot{x} + a_2(t)x = 0 \tag{27}$$

will be discussed. As known we do have here a state-space description

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^\bullet = \begin{bmatrix} 0 & 1 \\ -a_2(t) & -a_1(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{28}$$

with

$$x = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{29}$$

The transformation

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ p_{21}(t) & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{30}$$

yields

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}^\bullet = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{31}$$

with p_{21} solution of

$$\dot{p}_{21} = -p_{21}^2 - a_1(t)p_{21} - a_2(t) \tag{32}$$

and

$$\left.\begin{aligned} \lambda_1 &= p_{21} \\ \lambda_2 &= -p_{21} - a_1 \end{aligned}\right\} \tag{33}$$

For a time-invariant system, the solution of (32) reduces to a constant which satisfies the well-known characteristic equation.
Thus for a second order system the Riccati differential equation itself is the generalization of the characteristic equation. Although there is a lot of literature on the Riccati equation, it is usefull to consider the equation in $(p_{21}, \dot{p}_{21})$-space. This is sketched in figure 3 for the case that a_1 and a_2 are constants (time-invariant system). This figure clearly shows that there are two points of equilibrium p_1 and p_2 for which $\dot{p}_{21} = 0$.

One of these points is stable, while the other one is unstable. See figure 3.
As a consequense, for any arbitrarily initial value $p_{21}(0) \neq p_2$ the solution for (32) will be a function with a limit p_1 for $t \to \infty$. This is sketched in figure 4. So there exists an ordening in roots of the characteristic equation. Moreover, another typical aspect of singularities in the solution is known as finite escape time.
It is remarked that there also exists an uppertriangle transformation which brings (31) to a diagonal form. This transformation is given by

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & q \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \tag{34}$$

with $q(t)$ a solution of

$$\dot{q} = (\lambda_1 - \lambda_2)q + 1 \tag{35}$$

It is concluded that for equal dynamic eigenvalues (and thus equal eigenvalues for time-invariant systems) we have a time-varying upper diagonal matrix with $q(t) = t$ to obtain the diagonal form for the system.
As an example that time-varying systems also have an associated Riccati equation with a stable and an unstable solution, the Euler equation

$$\ddot{x} + 6t^{-1}\dot{x} + 6t^{-2}x = 0 \tag{36}$$

is considered. Thus the associated Riccati equation is given by

$$-p^2 - 6t^{-1}p - 6t^{-2} = \dot{p} \tag{37}$$

So we have

$$(tp)^{\bullet} = -\frac{1}{t}\left[(tp)^2 + 5(tp) + 6\right] \tag{38}$$

The equation yields as equilibrium solutions

$$tp_1 = -2 \quad \vee \quad tp_2 = -3 \tag{39}$$

If we assume that

$$p = \frac{-2 + \xi(t)}{t} \tag{40}$$

satisfies (38), then $\xi(t)$ has to satisfy

$$\dot{\xi} = -t^{-1}\xi - t^{-1}\xi^2 \tag{41}$$

Thus

$$\xi(t) = \frac{\xi(1)}{\{1 + \xi(1)\}t - \xi(1)} \tag{42}$$

and remark that the solution (42) yields

$$\lim_{t \to \infty} \xi(t) = 0 \tag{43}$$

So

$$p_1 = -2t^{-1} \tag{44}$$

represents a stable solution of (37).
If we assume, contrarily,

$$p = \frac{-3 + \eta(t)}{t} \tag{45}$$

as a solution of (38), then $\eta(t)$ can be calculated as

$$\eta(t) = \frac{\eta(1)t}{\eta(1)t - \eta(1) + 1} \tag{46}$$

and remark that solution (46) yields

$$\lim_{t \to \infty} \eta(t) = 1 \tag{47}$$

So the unstable solution is given by

$$p_2 = -3t^{-1} \tag{48}$$

5. CONCLUSION

Linear time-varying systems are discussed. It is stated how modal solutions can be constructed. This needs the triangularization of the state-space matrix of the system. To obtain zeros left of the diagonal in the state-space matrix gives for every row a set of Riccati equations. It is shown that for a third order system there are two sets of Riccati equations. For the more restricted linear time-invariant system they reduce to the characteristic equation. As a consequence properties of the characteristic equation are already contained in the properties of the Riccati equations. With the second order system, it is demonstrated that there is an ordening in the solution of the Riccati equation or the characteristic equation. Moreover it is demonstrated that the extension to time-varying systems removes the singular role of the Jordan form in time-invariant systems.

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