On Characteristic Equations, Dynamic Eigenvalues, Lyapunov Exponents and Floquet Numbers for Linear Time-Varying Systems

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Abstract

The meaning of linear time-varying systems for electronic circuits is sketched. It is seen that time-varying systems have a modal solution in terms of dynamic eigenvectors and dynamic eigenvalues. They reduce to the classical eigenvectors and eigenvalues in time-invariant systems. It is demonstrated that the Riccati equation takes the role of the characteristic equation. Moreover, the mean value of dynamic eigenvalues equals to the Lyapunov exponents and the Floquet numbers in the periodic case.

1 Introduction

Electronic circuits are nonlinear by nature. Roughly speaking, two different kinds of operation can be distinguished. The first one is described by the behavior of small signals around a fixed operating point, like class A amplifiers. The behavior of small signals can be derived as a set linear algebraic differential equations with constant coefficients. This set of equations is used for stability problems, distortion problems, noise problems and so on. In a mathematical sense, the set of equations is obtained by considering variations around the fixed operating point and hence known as the set of variational equations.
The second kind of operation contains circuits that behave in a time-varying mode of operation, like oscillators. Here also the set of variational quations is identified as a set of linear differential equations. The coefficients, however, are time-dependent. The field of applications is the same as in the first kind of operation: stability problems, distortion problems, noise problems and so on. The time-behavior of the coefficients in the variational equations is derived from the time-behavior of the (time-varying) mode of the circuit. For oscillators, the coefficients of the variational (differential) equations are periodic functions of time.
In [1] a representation for the solution of linear time-varying differential equations is derived, either in the form of the fundamental matrix or in the form of a sum of modal solutions. Moreover,

it is shown there that these modal solutions reduce to the well-known modal solutions of the exponential type for invariant set of equations. These modal solutions are characterized for circuits with n dynamical elements as the product of an n-dimensional dynamic eigenvector and an exponential function containing the dynamic eigenvalues.
For the subclass of linear time-varying differential equations with periodic coefficients the fundamental solution can be represented as the product of a periodic matrix and an exponential matrix containing the Floquet numbers [2]. As a consequence there are two representations for solutions of linear time-varying differential equations with periodic coefficients.

Since the solution is unique, there must be relations between the periodic matrix and Floquet numbers on one hand, and the dynamic eigenvectors and dynamic eigenvalues at the other. It turns out that the Floquet numbers are mean values of the dynamic eigenvalues. As a consequence dynamic eigenvalues contain more detailed information in comparison with the Floquet numbers. They are relevant in general stability problems [3] for nonlinear systems and they might give a theoretical base of moving poles in oscillator problems [4].

If the coefficients of the linear time-varying differential equations are non-periodic, then the Floquet numbers has to be replaced by the Lyapunov exponents. Thus, it appears to be usefull to include this aspect also is this paper.

The paper is divided is 7 sections. After this introduction, in section 2 is shown how modal solutions can be obtained. The fundamental solution of linear time-varying differential equation is also derived. For invariant systems the eigenvalues can be derived from a characteristic equation. In section 3 such an equation is derived for time-varying systems. More basic is, however, a Riccati differential equation, whose solution is also needed in solving a characteristic equation.

In the section 4 and 5 the proof is given that the Floquet numbers, the complex Lyapunov exponents [5] and the mean values of the dynamical eigenvalues are equal.

Section 6 gives an example how a modal solution is obtained. Finally, in section 7 some conclusions will be given.

2 The Modal Solution For A Linear Time-Varying System

Consider a homogeneous linear time-varying system

$$\dot{\boldsymbol{x}} = \mathbf{A}(t)\boldsymbol{x} \tag{2.1}$$

where $\boldsymbol{x} \in \mathbb{R}$ and $\mathbf{A}(t) \in \mathbb{R}^{n\times n}$ with as elements functions $a_{ij}(t)$ of time $t \in \mathbb{R}$ such that an unique solution of (2.1) is guaranteed. The system (2.1) will be transformed to a diagonal one in two steps. The first step brings (2.1) in a upper triangular form

$$\dot{\boldsymbol{y}} = \mathbf{B}(t)\boldsymbol{y} \tag{2.2}$$

This is achieved with a transformation

$$\boldsymbol{x} = \mathbf{R}(t)\boldsymbol{y} \tag{2.3}$$

As a result we must have that $\mathbf{A}(t)$ and $\mathbf{B}(t)$ are related according to

$$\mathbf{B}(t) = \mathbf{R}^{-1}(t)\mathbf{A}(t)\mathbf{R}(t) - \mathbf{R}^{-1}(t)\dot{\mathbf{R}}(t) \tag{2.4}$$

Note that the dynamic behaviour of the transformation (2.4) causes that $\mathbf{A}(t)$ and $\mathbf{B}(t)$ are not similar in the algebraic sense. The matrix $\mathbf{R}(t)$ is a product of transformations

$$\mathbf{R} = \mathbf{R}^{(n)} \ldots \mathbf{R}^{(2)} \tag{2.5}$$

with

$$\mathbf{R}^{(n-k)} = \begin{bmatrix} \mathbf{P}_{n-k}(t) & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_k \end{bmatrix}, \mathbf{P}_{n-k} = \begin{bmatrix} \mathbf{I}_{n-k-1} & \boldsymbol{0} \\ \boldsymbol{p}_{n-k}^T(t) & 1 \end{bmatrix} \tag{2.6}$$

and

$$\boldsymbol{p}_{n-k}^T = [p_{n-k,1} \ldots p_{n-k,n-k-1}] \tag{2.7}$$

It can be shown that each $\boldsymbol{p}_{n-k}^T$ has to satisfy a Riccati differential equation.
The first step

$$\boldsymbol{x} = \mathbf{R}^{(n)}(t)\boldsymbol{y}^{(n)} \tag{2.8}$$

will be treated in some detail.
For that purpose $\mathbf{A}$ is partitioned as

$$\mathbf{A} = \begin{bmatrix} \tilde{\mathbf{A}}_{n-1n-1} & \boldsymbol{a}_{12} \\ \boldsymbol{a}_{21}^T & a_{nn} \end{bmatrix} \tag{2.9}$$

We get

$$\dot{\boldsymbol{y}}^{(n)} = \begin{bmatrix} \tilde{\mathbf{A}}_{n-1n-1} + \boldsymbol{a}_{12}\boldsymbol{p}^T & \boldsymbol{a}_{12} \\ \boldsymbol{0}^T & \lambda_n \end{bmatrix} \boldsymbol{y}^{(n)} \tag{2.10}$$

iff $\boldsymbol{p}^T$ satisfies the Riccati equation

$$\dot{\boldsymbol{p}}^T = -\boldsymbol{p}^T\boldsymbol{a}_{12}\boldsymbol{p}^T - \boldsymbol{p}^T\tilde{\mathbf{A}}_{n-1n-1} + a_{nn}\boldsymbol{p}^T + \boldsymbol{a}_{21}^T \tag{2.11}$$

and if

$$\lambda_n = a_{nn} - \boldsymbol{p}^T\boldsymbol{a}_{12} \tag{2.12}$$

It is obvious how the procedure proceeds on the submatrix

$$\mathbf{A}_{n-1n-1} = \tilde{\mathbf{A}}_{n-1n-1} + \boldsymbol{a}_{12}\boldsymbol{p}^T \tag{2.13}$$

with $\mathbf{A}_{n-1n-1} \in \mathbb{R}^{(n-1)\times(n-1)}$.
Finally (2.2) will result with the matrix $\mathbf{B}(t)$ as

$$\mathbf{B}(t) = \begin{bmatrix} \lambda_1(t) & \ldots & b_{ij}(t) \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \lambda_n(t) \end{bmatrix} \tag{2.14}$$

The second step is to bring (2.1) into a diagonal form. The first manner to realize this, uses as a second transformation with an upper triangular matrix $\mathbf{Q}(t)$

$$\boldsymbol{y} = \mathbf{Q}(t)\boldsymbol{z} \tag{2.15}$$

so that there results a diagonal system

$$\dot{\boldsymbol{z}} = \mathrm{Diag}[\lambda_i(t)]\boldsymbol{z} \tag{2.16}$$

Thus

$$\boldsymbol{z}(t) = \mathrm{Diag}[e^{\gamma_i(t)}]\boldsymbol{z}(0) \tag{2.17}$$

with

$$\gamma_i(t) = \int\limits_0^t \lambda_i(\tau)d\tau \tag{2.18}$$

The solution of (2.1), then, can be written as

$$\boldsymbol{x}(t) = \mathbf{U}(t)\,\mathrm{Diag}[e^{\gamma_i(t)}]\mathbf{U}^{-1}(0)\boldsymbol{x}(0) \tag{2.19}$$

with

$$\mathbf{U}(t) = \mathbf{R}(t)\mathbf{Q}(t) \tag{2.20}$$

The second manner to realize the diagonal form from (2.2) with $\mathbf{B}(t)$ according to (2.14) starts with the assumption that

$$\boldsymbol{y}(t) = \mathbf{F}(t)\,\mathrm{Diag}[e^{\gamma_i(t)}]\boldsymbol{y}(0) \tag{2.21}$$

This is, indeed, a solution of (2.2) when $\mathbf{F}(t)$ is an upper triangular matrix with $f_{ii} = 1$ which has to satisfy

$$\dot{\mathbf{F}} + \mathbf{F}\,\mathrm{Diag}[\lambda_i(t)] = \mathbf{B}\mathbf{F} \tag{2.22}$$

$$\mathbf{F}(0) = \mathbf{1} \tag{2.23}$$

One can solve all $f_{ij}(i < j)$ from a set of first order uncoupled differetial equations With (2.3) and (2.21) we obtain

$$\boldsymbol{x}(t) = \mathbf{R}(t)\mathbf{F}(t)\,\mathrm{Diag}[e^{\gamma_i(t)}]\mathbf{R}^{-1}(0)\boldsymbol{x}(0) \tag{2.24}$$

Because the solution of (2.1) is unique, the representations (2.19) and (2.24) must yield the relation

$$\mathbf{U}(t) = \mathbf{R}(t)\mathbf{F}(t) \tag{2.25}$$

If

$$\mathbf{U}(t) = [\boldsymbol{u}_1(t), \ldots, \boldsymbol{u}_n(t)] \tag{2.26}$$

then (2.19) can be written as

$$\boldsymbol{x}(t) = \sum_{i=1}^{n} \boldsymbol{u}_i(t)e^{\gamma_i(t)}c_i \tag{2.27}$$

with c_i as a component of a vector $\boldsymbol{c}$ such that

$$\boldsymbol{c} = [c_1, \ldots, c_n]^T = \mathbf{U}^{-1}(0)\boldsymbol{x}(0) \tag{2.28}$$

Remark also that the fundamental solution $\boldsymbol{\Phi}(t, 0)$ of (2.1) is given by

$$\boldsymbol{\Phi}(t, 0) = \mathbf{U}(t)\,\mathrm{Diag}[e^{\gamma_1(t)}, \ldots, e^{\gamma_n(t)}]\mathbf{U}^{-1}(0) \tag{2.29}$$

Due to (2.27) the $\boldsymbol{u}_i(t)$ are called dynamic eigenvectors and $\lambda_i(t)$ are called dynamic eigenvalues.

3 Characteristic Equations of Linear Time-Varying Systems

In this section the relations (2.11) and (2.12) will be combined in order to formulate a kind of characteristic equation for time-varying systems. We start with the substitution of (2.12) into (2.11), yielding

$$\dot{\boldsymbol{p}}^T = -\boldsymbol{p}^T\tilde{\mathbf{A}}_{n-1n-1} + \lambda_n \boldsymbol{p}^T + \boldsymbol{a}_{21}^T \tag{3.1}$$

So, together with (2.12), we have

$$\begin{bmatrix}\boldsymbol{p}^T & -1\end{bmatrix}^{\boldsymbol{\cdot}} = \begin{bmatrix}\boldsymbol{p}^T & -1\end{bmatrix}\begin{bmatrix}\lambda_n - \tilde{\mathbf{A}}_{n-1n-1} & -\boldsymbol{a}_{12} \\ -\boldsymbol{a}_{21}^T & \lambda_n - a_{nn}\end{bmatrix} \tag{3.2}$$

Or in the original notation

$$\begin{bmatrix}\boldsymbol{p}^T & -1\end{bmatrix}^{\boldsymbol{\cdot}} = \begin{bmatrix}\boldsymbol{p}^T & -1\end{bmatrix}(\lambda_n \mathbf{I} - \mathbf{A}) \tag{3.3}$$

This is a generalization of the characteristic equation for time-invariant systems. To prove that point, remark that (2.11) allows constant solutions $\boldsymbol{p}^T$ for constant matrices $\mathbf{A}$. This means that the left hand side of (3.3) becomes zero for a constant system and we get a homogeneous linear set of equations in (3.3). Since $\begin{bmatrix}\boldsymbol{p}^T & -1\end{bmatrix} \neq \boldsymbol{0}^T$ we must have then as consequence

$$\det(\lambda_n - \mathbf{A}) = 0 \tag{3.4}$$

It is remarked in section 2 that the introduction of one row of zeros left of the diagonal in the n-th row of $\mathbf{A}(t)$ yields one characteristic equation. When this triangularization process as indicated in the preceding section, is applied in a second step to row $n-1$, we will get a second characteristic equation.

Totally, we get in a number of consecutive steps for making zeros left of the diagonal in the matrix $n-1$ characteristic equations of the type of (3.3). Because this is true ifor time-varying systems, it must also be true for time-invariant systems. There are has $n-1$ equations of the type (3.3), one of degree n, a second one of degree $n-1$ and a last one of degree 2. It can be proved [5] that for invariant systems the roots of the equation of degree $i < n$ are also roots of the equation of degree n. This confirms the classical approach for invariant systems to analyse only the highest degree equation.

Consider as an example the SISO time-varying system

$$\frac{d^n z}{dt^n} + a_1(t)\frac{d^{n-1}z}{dt^{n-1}} + \cdots + a_{n-1}(t)\frac{dz}{dt} + a_n(t)z = 0 \tag{3.5}$$

This equivalent to the state space description

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= -a_n(t)x_1 - a_{n-1}(t)x_2 - \cdots - a_1(t)x_n \end{aligned}\right\} \tag{3.6}$$

with the read out equation

$$z = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \tag{3.7}$$

For this situation the equation (3.1) reads

$$\dot{p}_i = -p_{i-1} + \lambda_n p_i - a_{n-i+1} \quad (i = 1, 2, \ldots, n-1) \tag{3.8}$$

and with (2.12) we have

$$\lambda_n = -p_{n-1} - a_1 \tag{3.9}$$

Writing the equation (3.8) as

$$\lambda_n^i p_i - \lambda_n^{i-1} p_{i-1} = (\dot{p}_i + a_{n-i+1})\lambda_n^{i-1} \quad (i = 1, 2, \ldots, n-1) \tag{3.10}$$

and adding them all together results in

$$\lambda_n^{n-1} p_{n-1} = \sum_{i=1}^{n-1} (\dot{p}_i + a_{n-i+1})\lambda_n^{i-1} \tag{3.11}$$

The use of (3.9) to eliminate p_{n-1} from the left hand side of (3.11) gives

$$\lambda_n^n + a_1\lambda_n^{n-1} + \sum_{j=2}^{n} (a_i + \dot{p}_{n-i+1})\lambda_n^{n-j} = 0 \tag{3.12}$$

It is not difficult to see that for invariant systems (3.12) is the well-known characteristic equation of (3.5). For time-varying systems, however, one first has to solve (3.8) , (3.9) and this automatically involves the solution of (3.12). So the Riccati equation is of fundamental interest for time-varying systems.

Moreover, as in the work of Riccati itself [6], also here the Riccati equation serves for order reduction. To show this we mention that (3.6) under the transformation (2.3) and (2.6) and with $\boldsymbol{p}^T = \begin{bmatrix} p_1 & \dots & p_{n-1} \end{bmatrix}$ according to (3.8) and (3.9) will yield

$$\left.\begin{aligned} \dot{y}_1^{(n)} &= y_2^{(n)} \\ &\vdots \\ \dot{y}_{n-2}^{(n)} &= y_{n-1}^{(n)} \\ \dot{y}_{n-1}^{(n)} &= p_1(t)y_1^{(n)} + \dots + p_{n-1}(t)y_{n-1}^{(n)} + y_n^{(n)} \end{aligned}\right\} \tag{3.13}$$

$$\dot{y}_n^{(n)} = \lambda_n y_n^{(n)} \tag{3.14}$$

Moreover we have

$$z = x_1 = y_1^{(n)} \tag{3.15}$$

Now one can derive with (3.13) and (3.15)

$$y_n^{(n)} = z^{(n-1)} - p_{n-1}z^{(n-2)} - \dots - p_1 z \tag{3.16}$$

This yields for (3.5), when using (3.15)

$$(D - \lambda_n)(z^{(n-1)} - p_{n-1}z^{(n-2)} - \dots - p_1 z) = 0 \tag{3.17}$$

It is seen with this elementary Floquet decomposition that with the solution $p_1, \dots, p_{n-1}$ of the Riccati equation the order of (3.5) is reduced. Moreover λ_n appears to be a characteristic quantity for the Floquet decomposition.

4 The Relation between Lyapunov Exponents and Dynamic Eigenvalues

The complete transformation of (2.1) into (2.16) shows that the aymptotoc behaviour of $\boldsymbol{x}(t)$ is governed by the integrals $\gamma_i(t)$ of the dynamic eigenvalues $\lambda_i(t)$. So there must be a relation with the Lyapunov exponents.

To describe this relation, we start with the fundamental solution $\boldsymbol{\Phi}_{\boldsymbol{\Lambda}}(t)$ of (2.17) The product $\boldsymbol{\Phi}_{\boldsymbol{\Lambda}}^H(t)\boldsymbol{\Phi}_{\boldsymbol{\Lambda}}(t)$ furnishes an ellipsoid whose principal axes are given by the singular values $\sigma_i(t)$ of $\boldsymbol{\Phi}_{\boldsymbol{\Lambda}}(t)$.
We thus have

$$\sigma_i^2(t) = e^{\overline{\gamma_i(t)}}e^{\gamma_i(t)} \tag{4.1}$$

So

$$\mathrm{Re}[\gamma_i(t)] = \ln \sigma_i(t) \tag{4.2}$$

According to [7] the Lyapunov exponent χ_i is defined by

$$\chi_i = \lim_{t\to\infty} \frac{1}{t} \ln \sigma_i(t) \tag{4.3}$$

This has as a consequence that the Lyapunov exponent is to be interpretated as an average of the real part of a dynamic eigenvalue over a sufficient long period of time.

It is clear that we are in a position to define a complex Lyapunov exponent by (compare [8])

$$L_i = \lim_{t\to\infty} \frac{1}{t} \int_0^t \lambda_i(t) d\tau \tag{4.4}$$

So one has

$$\chi_i = \mathrm{Re}(L_i) \tag{4.5}$$

A second definition for Lyapunov exponents uses the asymptotic behavior of functions. To show the equivalence with the given definition, write

$$e^{\gamma_k(t)} = e^{\{\gamma_k(t) - L_k t\}} e^{(L_k - L_1)t} e^{L_1 t} \tag{4.6}$$

If

$$\chi_1 = \mathrm{Re}(L_1) > \mathrm{Re}(L_k) = \chi_k \quad (k = 2, 3, \ldots, n) \tag{4.7}$$

Then

$$\lim_{t\to\infty} e^{(L_k - L_1)t} = 0 \tag{4.8}$$

$$\lim_{t\to\infty} e^{\{\gamma_k(t) - L_k t\}} = 1 \tag{4.9}$$

So we will have

$$x_i(t) \longrightarrow \hat{x}_i(t) e^{L_1 t} \quad \text{if} \quad t \to \infty \tag{4.10}$$

for all components of $\boldsymbol{x}(t)$.

One now directly concludes from (4.10)

$$\lim_{t\to\infty} \frac{1}{t} \ln |x_i(t)| = \lim_{t\to\infty} \frac{1}{t} \ln |\hat{x}_i(t)| + \chi_1 \tag{4.11}$$

Thus

$$\chi_1 = \lim_{t\to\infty} \frac{1}{t} \ln |x_i(t)| \tag{4.12}$$

if the amplitude function $\hat{x}_i(t)$ has a regular character.

5 Linear Systems with Periodic Coefficients

In this section we consider (2.1) under the condition that $\mathbf{A}(t)$ is periodic, thus

$$\mathbf{A}(t) = \mathbf{A}(t + T) \tag{5.1}$$

As a consequence also (2.11) has periodic coefficients, so that one searches for Riccati with periodic solutions [9]. If they exist then (2.12) gives that the dynamic eigenvalues are periodic. In that case the relation between the dynamic eigenvalues and the Floquet numbers may be questioned.

If $\lambda_i(t)$ is periodic with period T, then the complex Lyapunov exponent L_i is, according to (4.4), given by

$$L_i = \frac{1}{T}\int_0^T \lambda_i(\tau)d\tau \tag{5.2}$$

And we can write

$$\gamma_i(t) = \{\gamma_i(t) - L_i t\} + L_i t \tag{5.3}$$

Remark that $\{\gamma_i(t) - L_i t\}$ will be a periodic function if $\lambda_i(t)$ is, while $L_i t$ is linear.
As a consequence the matrix $\mathrm{Diag}[e^{\gamma_i(t)}]$ can be written as the product of a periodic matrix and a second one with linear argument

$$\mathrm{Diag}[e^{\gamma_i(t)}] = \mathrm{Diag}[e^{\{\gamma_i(t)-L_i t\}}]\,\mathrm{Diag}[e^{L_i t}] \tag{5.4}$$

This means that we can write the solution (2.19) as

$$\boldsymbol{x}(t) = \mathbf{P}(t)e^{\mathbf{S}t}\boldsymbol{x}(0) \tag{5.5}$$

with

$$\mathbf{P}(t) = \mathbf{U}(t)\,\mathrm{Diag}[\gamma_i(t) - L_i t]\mathbf{U}^{-1}(0) \tag{5.6}$$

$$\mathbf{S} = \mathbf{U}(0)\,\mathrm{Diag}[L_i]\mathbf{U}^{-1}(0) \tag{5.7}$$

If the matrix $\mathbf{U}(t)$ containing the dynamic eigenvectors is periodic (which depends on the periodicity of the solutions of the Riccati equations) then $\mathbf{P}(t)$ in (5.5) will be periodic and $\mathbf{S}$ will be a constant matrix. So (5.5) will be the Floquet representation of (2.1) under the condition (5.1). The Floquet numbers are the eigenvalues of $\mathbf{S}$. With (5.7) it is shown that the Lyapunov exponents and the mean value of the dynamic eigenvalues are the Floquet numbers.

6 Example

In this section an example is presented in order to show the solution procedure indicated in section 2. Moreover, it is demonstrated how solutions of the Riccati equations give dynamic eigenvalues. From them the Floquet numbers can be calculated. The example is presented in [10] and originates from [11]

$$\left.\begin{aligned}\dot{x}_1 &= (-1 - 9\cos^2 6t + 12\sin t\cos t)x_1 + (12\cos^2 6t + 9\sin t\cos t)x_2\\ \dot{x}_2 &= (-12\sin^2 6t + 9\sin t\cos t)x_1 + (1 + 9\sin^2 6t + 12\sin t\cos t)x_2\end{aligned}\right\} \tag{6.1}$$

This is simplified by introducing

$$\tau = 6t + \frac{1}{2}\phi \tag{6.2}$$

with

$$\cos\phi = \frac{3}{5} \quad , \quad \sin\phi = \frac{4}{5} \tag{6.3}$$

So

$$\left.\begin{aligned}\frac{dx_1}{d\tau} &= (\frac{1}{3}-\frac{5}{2}\cos^2\tau)x_1 + (1+\frac{5}{2}\sin\tau\cos\tau)x_2\\ \frac{dx_2}{d\tau} &= (-1+\frac{5}{2}\sin\tau\cos\tau)x_1 + (\frac{1}{3}-\frac{5}{2}\sin^2\tau)x_2\end{aligned}\right\} \tag{6.4}$$

This equals to be a special case of the example in [12], page 113. Transform according to (2.3) for $n = 2$ as

$$\left.\begin{aligned}x_1 &= y_1\\ x_2 &= p(\tau)y_1 + y_2\end{aligned}\right\} \tag{6.5}$$

Then the Riccati equation (2.11) is a scalar equation for this second order system and reads

$$\frac{dp}{d\tau} = -(1+\frac{5}{2}\sin\tau\cos\tau)p^2 - \frac{5}{2}(\sin^2\tau - cos^2\tau)p + +(-1+\frac{5}{2}\sin\tau\cos\tau) \tag{6.6}$$

Inspection gives as a periodic solution

$$p = -\tan(\tau) \tag{6.7}$$

So the equation (2.2) becomes now

$$\left.\begin{aligned}\frac{dy_1}{d\tau} &= (-\frac{13}{6}-\tan\tau)y_1 + (1+\frac{5}{2}\sin\tau\cos\tau)y_2\\ \frac{dy_2}{d\tau} &= \qquad\qquad\qquad (\frac{1}{3}+\tan\tau)y_2\end{aligned}\right\} \tag{6.8}$$

Using the transformation

$$\left.\begin{aligned}y_1 &= z_1 + q(\tau)z_2\\ y_2 &= \qquad z_2\end{aligned}\right\} \tag{6.9}$$

we can bring the set to the diagonal form

$$\left.\begin{aligned}\frac{dz_1}{d\tau} &= (-\frac{13}{6}-\tan\tau)z_1\\ \frac{dz_2}{d\tau} &= (\frac{1}{3}+\tan\tau)z_2\end{aligned}\right\} \tag{6.10}$$

iff $q(\tau)$ satisfies

$$\frac{dq}{d\tau} = -\frac{5}{2}q - 2\tan(\tau)q + 1 + \frac{5}{2}\sin\tau\cos\tau \tag{6.11}$$

with a particular solution

$$q(\tau) = \sin\tau\cos\tau \tag{6.12}$$

We now arrive at

$$\begin{bmatrix}x_1(\tau)\\ x_2(\tau)\end{bmatrix} = \begin{bmatrix}1 & 0\\ -\tan\tau & 1\end{bmatrix}\begin{bmatrix}1 & \sin\tau\cos\tau\\ 0 & 1\end{bmatrix}\begin{bmatrix}z_1(\tau)\\ z_2(\tau)\end{bmatrix} \tag{6.13}$$

with as solutions of (6.10)

$$\begin{bmatrix} z_1(\tau) \\ z_2(\tau) \end{bmatrix} = \begin{bmatrix} \cos\tau & 0 \\ 0 & \frac{1}{\cos\tau} \end{bmatrix} \begin{bmatrix} e^{-\frac{13}{6}\tau} & 0 \\ 0 & e^{\frac{1}{3}\tau} \end{bmatrix} \begin{bmatrix} z_1(0) \\ z_2(0) \end{bmatrix} \tag{6.14}$$

Thus

$$\begin{bmatrix} x_1(\tau) \\ x_2(\tau) \end{bmatrix} = \begin{bmatrix} \cos\tau & \sin\tau \\ -\sin\tau & \cos\tau \end{bmatrix} \begin{bmatrix} e^{-\frac{13}{6}\tau} & 0 \\ 0 & e^{\frac{1}{3}\tau} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \tag{6.15}$$

It is remarked that the dynamic eigenvalues are given by

$$\lambda_1(\tau) = -\frac{13}{6} - \tan\tau \quad , \quad \lambda_2(\tau) = \frac{1}{3} + \tan\tau \tag{6.16}$$

with

$$\chi_1 < 0 \quad , \quad \chi_2 > 0 \tag{6.17}$$

The system is thus unstable. Should one, however, determine the classical eigenvalues of the system matrix of (6.3), (6.4), then one find that both are negative ($-\frac{5}{3}$ and $-\frac{1}{6}$). This is a simple confirmation that eigenvalues have no meaning for time-varying systems.

7 Conclusions

It is demonstrated that an n-th order linear time-varying system can be characterized by n dynamic eigenvalues and n dynamic eigenvectors. Each of these pairs can be derived with the solution of a differential equation of Riccati. This Riccati equation also can be used to derive a characteristic equation for the dynamic eigenvalues. So there is a set of $n-1$ charcateristic equations for an n-th order system. In [5], it is shown that this is exactly the same as for invariant systems. The solution of lower order equations in the invariant case is already part of the solution for the higher order equation.

This does not hold for general linear time-varying systems. There the Riccati equation is more basic than the characteristic equation. Further it is demonstrated that the mean value of a dynamic eigenvalue equals a newly introduced complex Lyapunov exponent, which for periodic systems reduces to a Floquet number. As a consequence dynamic eigenvalues contain more information about the sytem than the Lyapunov exponents and Floquet numbers. The last two play a role only in the asymptotic behavior.

Finally, it is demonstrated for a second order system how a modal solution can be constructed. The result shows that classical eigenvalues give wrong impressions on the behavior of time-varying systems.

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