# Schemes of Polynomial Characteristic Equations for Scalar Linear Systems

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**Abstract**

General scalar linear systems are addressed. It is shown that the earlier introduced dynamic eigenvalues satisfy a scheme of polynomial characteristic equations of decreasing order.

# 1 Introduction

As is well known, small variations around a particular solution trajectory of an arbitrarily nonlinear dynamical system satisfy a linear time-varying (LTV) equation [1]. As was shown earlier, the modal solutions of this type of equations turned out to be fully characterized by the so-called dynamic eigenvalues [2]. Moreover, the Riccati equation was recognized as the characteristic equation [3, 4].

In this article, it is proven that the dynamic eigenvalues alternatively satisfy a scheme of polynomial equations of decreasing order. In this scheme, the time-dependent coefficients of each lower order polynomial equation incorperate the dynamic eigenvalue solutions of all higher order polynomial equations.

At first glance this seems to be in contradiction with the theory of linear time-invariant (LTI) systems with precisely one characteristic equation for the complete eigenspectrum. However, in section 2 it will be shown that also for the familiar LTI systems there is a scheme of characteristic equations corresponding to the set of eigenvalues.

In section 3, the state-space approach for LTV systems is used in order to obtain the dynamic eigenvalues and the corresponding characteristic equation [5]. The results are in agreement with LTI systems. It is demonstrated that the scheme of characteristic equations for LTV systems cannot be reduced to a single one, as is the case for LTI systems.

Finally, it is shown how the Cauchy-Floquet decomposition can be obtained without using the state-space approach (compare [6]).

## 2 A Scheme of Characteristic Equations for LTI Systems

Assume that the homogeneous scalar input-output relation for LTI systems is given by

$$a_0 D^n x + a_1 D^{n-1} x + \cdots + a_{n-1} D x + a_n x = 0\,, \tag{2.1}$$

where $D^k$ denotes the $k$-th derivative to the time $t$ and $x = x(t)$ the output signal, respectively, while $a_0, a_1, \ldots, a_n$ are constant system parameters, or, more compactly

$$\sum_{i=0}^{n} a_i D^{n-i} x(t) = 0\,. \tag{2.2}$$

In the next, relation (2.1) and (2.2) will be normalized by setting

$$a_0 = 1\,. \tag{2.3}$$

If we write for the time derivatives

$$D^k x = D^{k-1}[D - \lambda]x + \lambda D^{k-1} x\,, \tag{2.4}$$

where $\lambda$ denotes some constant, relation (2.2) can be expanded as

$$\sum_{i=0}^{n-1} \alpha_i D^{n-1-i}[D-\lambda]x(t) + \sum_{i=0}^{n} a_i \lambda^{n-i} x(t) = 0\,, \tag{2.5}$$

in which the constant coefficients $\alpha_i$ are given by

$$\alpha_i = \sum_{j=0}^{i} a_j \lambda^{i-j} \quad (i = 0, 1, \ldots, n-1)\,. \tag{2.6}$$

Equation (2.5) shows that a modal solution of the form

$$x = \exp(\lambda t) \tag{2.7}$$

satisfies (2.1) or (2.2) if and only if the eigenvalue $\lambda = \lambda_n$ is a solution of the polynomial equation

$$\sum_{i=0}^{n} a_i \lambda_n^{n-i} = 0\,, \tag{2.8}$$

which at this place is called the *first characteristic equation*.
In a next expansion, for some constant $\mu$, equation (2.5) goes into

$$\sum_{i=0}^{n-2} \beta_i D^{n-2-i}[D-\mu]y(t) + \sum_{i=0}^{n-1} \alpha_i \mu^{n-1-i} y(t) + \sum_{i=0}^{n} a_i \lambda^{n-i} x(t) = 0 \tag{2.9}$$

in which

$$y(t) = [D - \lambda]x(t)\,, \tag{2.10}$$

while the new constant coefficients $\beta_i$ are given by

$$\beta_i = \sum_{j=0}^{i} \alpha_j \mu^{i-j} \quad (i = 0, 1, \ldots, n-2)\,. \tag{2.11}$$

Next, with

$$x = \exp(\mu t)\,, \tag{2.12}$$

we deduce from (2.8), (2.9) and with $\lambda = \lambda_n$, that even for equal eigenvalues, expression (2.12) is a modal solution of (2.1) or (2.2) if and only if $\mu = \lambda_{n-1}$ is a solution of the so-called *second characteristic equation*

$$\sum_{i=0}^{n-1} \alpha_i \lambda_{n-1}^{n-1-i} = 0\,. \tag{2.13}$$

Moreover, since

$$\sum_{i=0}^{n-1} \alpha_i \lambda_{n-1}^{n-1-i}[\lambda_{n-1} - \lambda_n] + \sum_{i=0}^{n} a_i \lambda_n^{n-i} = 0\,, \tag{2.14}$$

it follows by substitution of (2.6) for $\alpha_i$

$$\begin{aligned}\sum_{i=0}^{n-1} \alpha_i \lambda_{n-1}^{n-1-i}[\lambda_{n-1} - \lambda_n] + \sum_{i=0}^{n} a_i \lambda_n^{n-i} = \sum_{i=0}^{n} a_i \lambda_n^{n-i} - \sum_{i=0}^{n-1} a_j \lambda_n^{n-j} + \sum_{j=0}^{0} a_j \lambda_n^{0-j} \lambda_{n-1}^{n} + \\ + \sum_{j=0}^{1} a_j \lambda_n^{1-j} - \sum_{j=0}^{0} a_j \lambda_n^{1-j} \lambda_{n-1}^{n-1} + \cdots + \sum_{j=0}^{n-1} a_j \lambda_n^{n-1-j} - \sum_{j=0}^{n-1} a_j \lambda_n^{n-1-j} \lambda_{n-1}^{1} = \\ a_0 \lambda_{n-1}^{n} + a_1 \lambda_{n-1}^{n-1} + \cdots + a_{n-1} \lambda_{n-1}^{1} + a_n = 0\,.\end{aligned} \tag{2.15}$$

Thus, for LTI systems the first characteristic equation implies the second one.
From another point of view, equation (2.8) corresponds to an algebraic polynomial of degree $n$, while (2.13) shows an algebraic polynomial of degree $n-1$.
Furthermore, relation (2.6) directly implies for $i = 1, 2, \ldots, n-1$

$$\lambda \alpha_{n-i} = -a_{n-i+1} + \alpha_{n-i+1} \tag{2.16}$$

and for $i = 0$

$$a_0 \lambda = -a_1 + \alpha_1 \quad \text{with} \quad a_0 = 1\,. \tag{2.17}$$

Now, if $\lambda$ is eliminated from (2.16) and (2.17), we obtain for $i = 1, 2, \ldots, n-1$

$$(\alpha_1 - a_1)\alpha_{n-i} - \alpha_{n-i+1} + a_{n-i+1} = 0\,. \tag{2.18}$$

Next, we introduce the row vectors

$$\left.\begin{aligned} \boldsymbol{\alpha}^T &= [\,\alpha_{n-1}, \ldots, \alpha_1\,] \\ \boldsymbol{a}^T &= [\,a_n, \ldots, a_2\,] \\ \boldsymbol{e}_{n-1}^T &= [\,0, \ldots, 0, 1\,] \end{aligned}\right\} \tag{2.19}$$

in which $T$ stands for the transpose, and the shift-matrix $\mathbf{I}_{n-1}^+$ as

$$\mathbf{I}_{n-1}^+ = \begin{bmatrix} 0 & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \ldots & \ldots & 0 \end{bmatrix} . \tag{2.20}$$

As a consequence, (2.18) can be written as the vector algebraic Riccati equation [7]

$$\boldsymbol{\alpha}^T \boldsymbol{e}_{n-1} \boldsymbol{\alpha}^T - a_1 \boldsymbol{\alpha}^T - \boldsymbol{\alpha}^T \mathbf{I}_{n-1}^+ + \boldsymbol{a}^T = \boldsymbol{0}^T . \tag{2.21}$$

In the same way, the characteristic equation (2.13) together with (2.11) induces a second algebraic Riccati equation, namely

$$\boldsymbol{\beta}^T \boldsymbol{e}_{n-2} \boldsymbol{\beta}^T - b_1 \boldsymbol{\beta}^T - \boldsymbol{\beta}^T \mathbf{I}_{n-2}^+ + \boldsymbol{b}^T = \boldsymbol{0}^T , \tag{2.22}$$

with

$$\boldsymbol{\beta}^T = [\,\beta_{n-2}, \ldots, \beta_1\,], \boldsymbol{b}^T = [\,\alpha_{n-1}, \ldots, \alpha_2\,], b_1 = \alpha_1 . \tag{2.23}$$

Obviously, this process can be continued $(n-1)$ times. The final result is that the original input-output equation (2.1) or (2.2) is replaced by

$$\begin{aligned} &[D-\lambda_1][\ldots][D-\lambda_n]x + \sum_{i=0}^{1} \alpha_i^{(n-1)} \lambda_1^{1-i} [D-\lambda_2][\ldots] \times [D-\lambda_n]x + \\ &+ \sum_{i=0}^{2} \alpha_i^{(n-2)} \lambda_2^{2-i} [D-\lambda_3][\ldots][D-\lambda_n]x + \cdots + \sum_{i=0}^{n-1} \alpha_i^{(1)} \lambda_{n-1}^{n-1-i} [D-\lambda_n]x + \sum_{i=0}^{n} \alpha_i^{(0)} \lambda_n^{n-i} x = 0 . \end{aligned} \tag{2.24}$$

It is concluded that the original differential polynomial with constant coefficients in (2.1) or (2.2) is factorized from the right in terms of the eigenvalues $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$. The coefficients $\alpha_i^{(j)}$ $(j = 0, 1, \ldots, n-1)$ in (2.24) are obtained as

$$\alpha_i^{(0)} = a_i \; (i = 0, 1, \ldots, n) , \tag{2.25}$$

$$\alpha_i^{(j)} = \sum_{k=0}^{i} \alpha_k^{(j-1)} \lambda_j^{n+1-j} \text{for} \begin{cases} (i = 0, 1, \ldots, n-j) \\ (j = 1, 2, \ldots, n-1) , \end{cases} \tag{2.26}$$

with

$$\alpha_0^{(j)} = 1 \,. \tag{2.27}$$

Elimination of the eigenvalues $\lambda_j$ from (2.26) with $i = 1$ leads to

$$\alpha_1^{(j)} = \alpha_1^{(j-1)} + \lambda_j \tag{2.28}$$

which on its turn yields on account of (2.27) a set algebraic Riccati equations of a lower dimension.

# 3 A Scheme of Characteristic Equations for LTV-Systems

In the preceding section a scheme of polynomial characteristic equations for a LTI system has been derived. In it, each polynomial equation corresponds to a single algebraic Riccati equation.
In this section, first the reverse problem will be considered, that is, the Riccati equation will be obtained directly from the differential input-output equation, and afterwards the polynomial characteristic equation from the Riccati equation. For that purpose, the input-output equation (2.1) is rewritten in state-space description as

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \mathbf{I}_{n-1}^{+} & \boldsymbol{e}_{n-1} \\ -\boldsymbol{a}^T & -a_1 \end{bmatrix} \boldsymbol{x} \,, \tag{3.29}$$

where the dot stands for a differentiation with respect to the time $t$. This equation will be transformed to an alternative state-space description according to the transformation [8]

$$\boldsymbol{x} = \begin{bmatrix} \mathbf{I}_{n-1} & \boldsymbol{0} \\ \boldsymbol{p}^T & 1 \end{bmatrix} \boldsymbol{y} \,, \tag{3.30}$$

in which

$$\boldsymbol{p}^T = [p_1, \ldots, p_{n-1}] \,. \tag{3.31}$$

The result of this transformation can be stated as

$$\dot{\boldsymbol{y}} = \begin{bmatrix} \mathbf{I}_{n-1}^{+} + \boldsymbol{e}_{n-1}\boldsymbol{p}^T & \boldsymbol{e}_{n-1} \\ \boldsymbol{0}^T & \tilde{\lambda}_n \end{bmatrix} \boldsymbol{y} \,, \tag{3.32}$$

where

$$\tilde{\lambda}_n = -a_1 - p_{n-1} \tag{3.33}$$

and $\boldsymbol{p}^T$ satisfies the vector Riccati differential equation [7]

$$\dot{\boldsymbol{p}}^T = -\boldsymbol{p}^T \mathbf{I}_{n-1}^{+} - \boldsymbol{a}^T + \tilde{\lambda}_n \boldsymbol{p}^T \,. \tag{3.34}$$

It may be clear that we have assumed that the collumn vector $\boldsymbol{p}$ in (3.30) is a function of time, thus $\boldsymbol{p}^T = \boldsymbol{p}^T(t)$. This allows a generalization to LTV systems.
If $\boldsymbol{p}^T$ is assumed to be a constant, then the lefthand side of (3.34) reduces to zero and we have an

algebraic Riccati equation. In that case, it follows that $\tilde{\lambda}_n$ is a classical eigenvalue of the system given by (3.32) and thus of system (3.29). As a consequence, $\tilde{\lambda}_n$ is an eigenvalue of the original LTI system, given by (2.1) or (2.2).

Next, we have to show that the dynamic eigenvalue [2] $\tilde{\lambda}_n = \tilde{\lambda}_n(t)$ satisfies a polynomial characteristic equation for a scalar LTV system with input-output equation (2.1) in which the system parameters are time-varying , thus $a_i = a_i(t)$. For that purpose (3.34) is rewritten as

$$-p_i + \tilde{\lambda}_n p_{i+1} = a_{n-i} + \dot{p}_{i+1} \; (i = 0, 1, \ldots, n-2) \tag{3.35}$$

with $a_i = a_i(t)$ and $p_0 = 0$. If the equations in (3.35) are multiplied by $\tilde{\lambda}_n^i$ and subsequently added together, we obtain

$$\tilde{\lambda}_n^{n-1} p_{n-1} = \sum_{i=2}^{n} (a_i + \dot{p}_{n-i+1}) \tilde{\lambda}_n^{n-i} \, . \tag{3.36}$$

Elimination of $p_{n-1} = p_{n-1}(t)$ from (3.35) with the aid of (3.33) yields

$$\sum_{i=1}^{n} \bar{a}_i(t) \tilde{\lambda}_n^{n-i}(t) = 0 \, , \tag{3.37}$$

in which the modified polynomial time-dependent coefficients $\bar{a}_i(t)$ are given by

$$\bar{a}_i = a_i + \dot{p}_{n-i+1} \, , \tag{3.38}$$

with $p_n = 0$.

Note that for LTI systems, where $\boldsymbol{p}^T$ is a constant vector, and as a consequence $\dot{p}_{n-i+1} = 0$, equation (3.37) reduces to the classical characteristic polynomial equation with $\tilde{\lambda}_n$ an eigenvalue of the input-output equation (2.1) with constant system parameters.

Next, we show

$$\tilde{\lambda}_n = \lambda_1 \, . \tag{3.39}$$

To that aim, we remark that (3.32) yields

$$\left.\begin{aligned} \dot{y}_i &= y_{i+1} \quad (i = 1, 2, \ldots, n-2) \\ \dot{y}_{n-1} &= p_1 y_1 + \cdots + p_{n-1} y_{n-1} + y_n \\ \dot{y}_n &= \tilde{\lambda}_n y_n \end{aligned}\right\} . \tag{3.40}$$

As a consequence, we have

$$\left.\begin{aligned} D^{n-1} y_1 - p_{n-1} D^{n-2} y_1 - \ldots \\ \cdots - p_2 D y_1 - p_1 y_1 = y_n \\ \dot{y}_n = \tilde{\lambda}_n y_n \end{aligned}\right\} . \tag{3.41}$$

Hence, $y_n$ has the modal form [9]

$$y_n(t) = C \exp[\int^{t} \tilde{\lambda}_n(\tau) d\tau] \, , \tag{3.42}$$

with $C$ a constant. In addition, we have

$$[D-\tilde{\lambda}_n](D^{n-1}-p_{n-1}D^{n-2}-\cdots-p_2D-p_1)y_1=0\,. \tag{3.43}$$

It is observed that the original differential polynomial in (2.1) this time will be factorized from the left. Since transformation (3.30) implies

$$y_1=x_1=x\,, \tag{3.44}$$

equation (3.43) directly results into the identity (3.39).
It should be remarked, again, that (3.43) remains valid if the coefficients $a_i$ are functions of time, thus $a_i=a_i(t)$. To show this directly without the use of any state-space description, write

$$D^nx=[D-\tilde{\lambda}_n]D^{n-1}x+\tilde{\lambda}_nD^{n-1}x \tag{3.45}$$

with $\tilde{\lambda}_n=\tilde{\lambda}_n(t)$ and substitute (3.33), resulting into

$$D^nx=[D-\tilde{\lambda}_n]D^{n-1}x-(a_1+p_{n-1})D^{n-1}x\,, \tag{3.46}$$

with $a_i=a_i(t)$. As a consequence, we obtain the identity

$$D^nx+a_1D^{n-1}x=[D-\tilde{\lambda}_n]D^{n-1}x-p_{n-1}D^{n-1}x\,. \tag{3.47}$$

Next in (3.47) we apply

$$p_{n-1}D^{n-1}x=D(p_{n-1}D^{n-2}x)-\dot{p}_{n-1}D^{n-2}x=[D-\tilde{\lambda}_n]p_{n-1}D^{n-2}x+(\tilde{\lambda}_np_{n-1}-\dot{p}_{n-2})D^{n-2}x\,, \tag{3.48}$$

and subsequently use expression (3.35) for $i=n-2$. This yields

$$D^nx+a_1D^{n-1}x+a_2D^{n-2}x=[D-\tilde{\lambda}_n](D^{n-1}x-p_{n-1}D^{n-2}x)-p_{n-2}D^{n-2}x\,. \tag{3.49}$$

By repetition of the above arguments we get

$$\begin{aligned}D^nx+a_1D^{n-1}x+a_2D^{n-2}x+\cdots+a_{n-1}Dx=&\\ [D-\tilde{\lambda}_n](D^{n-1}x-p_{n-1}D^{n-2}x-\cdots-p_2Dx)-p_1Dx\,.&\end{aligned} \tag{3.50}$$

Finally, with

$$p_1Dx=[D-\tilde{\lambda}_n]p_1x+a_nx\,, \tag{3.51}$$

we arrive at

$$\begin{aligned}D^nx+a_1D^{n-1}x+a_2D^{n-2}x+\cdots+a_{n-1}Dx+a_nx=&\\ [D-\tilde{\lambda}_n](D^{n-1}x-p_{n-1}D^{n-2}x-\cdots-p_2Dx-p_1x)\,,&\end{aligned} \tag{3.52}$$

for $a_i=a_i(t)$. It is clear that this process can be continued until the Cauchy-Floquet decomposition is obtained.

# 4 Conclusions

It is proven that for linear time-invariant (LTI) as well as for time-varying (LTV) systems the differential system operator induces a scheme of coupled characteristic polynomials. In it, each polynomial equation corresponds to a single Riccati equation. Also, the coefficients of each lower order polynomial contain the solutions of all higher order polynomial equations.
For constant (LTI) systems, the scheme of coupled polynomial equations reduces to a single characteristic equation for the complete eigenspectrum. This is *not* the case for time-varying systems.

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