Floquet Numbers and Dynamic Eigenvalues

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Abstract—*The relation between Floquet numbers and dynamic eigenvalues is derived.*

I. Introduction

Electronic circuits are nonlinear by nature. Roughly speaking, two different kinds of operation can be distinguished. The first one is described by the behavior of small signals around a fixed operating point, like class A amplifiers. The behavior of small signals can be derived as a set linear algebraic differential equations with constant coefficients. This set of equations is used for stability problems, distortion problems, noise problems and so on. In a mathematical sense, the set of equations is obtained by considering variations around the fixed operating point and hence known as the set of variational equations.
The second kind of operation contains circuits that behave in a time-varying mode of operation, like oscillators. Here also the set of variational equations is identified as a set of linear differential equations. The coefficients, however, are time-dependent. The field of applications is the same as in the first kind of operation: stability problems, distortion problems, noise problems and so on. The time-behavior of the coefficients in the variational equations is derived from the time-behavior of the (time-varying) mode of the circuit. For oscillators, the coefficients of the variational (differential) equations are periodic functions of time. In [1] a representation for the solution of linear time-varying differential equations is derived, either in the form of the fundamental matrix or in the form of a sum of modal solutions. Moreover, it is shown there that these modal solutions reduce to the well-known modal solutions of the exponential type for invariant sets of equations. These modal solutions are characterized for circuits with n dynamical elements as the product of a n-dimensional dynamic eigenvector and an exponential function containing the dynamic eigenvalues.
For the subclass of linear time-varying differential equations with periodic coefficients the fundamental solution can also be represented as the product of a periodic matrix and an exponential matrix containing the Floquet numbers [2]. As a consequence there are two representations for solutions of linear time-varying differential equations with periodic coefficients.

Since the solution is unique, there must be relations between the periodic matrix and Floquet numbers on one hand, and the dynamic eigenvectors and dynamic eigenvalues on the other. It turns out that the Floquet numbers are mean values of the dynamic eigenvalues. As a consequence dynamic eigenvalues contain more detailed information in comparison with the Floquet numbers. They are relevant in general stability problems [3] for nonlinear systems and they might give a theoretical base of moving poles in oscillator problems [4].

The paper is divided is 5 sections. After this introduction, in section 2 is shown how modal solutions can be obtained for second order systems. In section 3 two examples, both with periodic coeficients are discussed. In both examples, the dynamic eigenvalues are collected in a diagonal matrix. This formulation deviates from the Floquet representations in both cases. It is shown in this section how these deviations can be suppressed. In section 4 some remarks with respect to the equivalence Floquet numbers and the mean value of dynamic eigenvalues is discussed. In section 5 some conclusions are formulated.

II. The Riccati Equation For A Second Order System

In this section a second order system is treated. It is shown how a diagonalization process is derived , involving two transformations. Both contain one unknown function. Consider

$$\left.\begin{aligned}\dot{x}_1 &= a_{11}(t)x_1 + a_{12}(t)x_2\\ \dot{x}_2 &= a_{21}(t)x_1 + a_{22}(t)x_2\end{aligned}\right\} \rightleftarrows \dot{\boldsymbol{x}} = \mathbf{A}\boldsymbol{x} \qquad (1)$$

Apply a first transformation

$$\left.\begin{aligned}x_1 &= y_1\\ x_2 &= p_1(t)y_1 + y_2\end{aligned}\right\} \rightleftarrows \boldsymbol{x} = \mathbf{P}_1\boldsymbol{y} \qquad (2)$$

If p_1 satisfies

$$\dot{p} = -a_{12}p^2 - (a_{11} - a_{22})p + a_{21} \tag{3}$$

then we get

$$\left.\begin{aligned} \dot{y}_1 &= \lambda_1(t,p_1)y_1 + a_{12}y_2 \\ \dot{y}_2 &= \lambda_2(t,p_1)y_2 \end{aligned}\right\} \rightleftarrows \dot{\boldsymbol{y}} = \mathbf{B}_1\boldsymbol{y} \tag{4}$$

Here

$$\left.\begin{aligned} \lambda_1(t,p_1) &= a_{11} + a_{12}p_1 \\ \lambda_2(t,p_1) &= a_{22} - a_{12}p_1 \end{aligned}\right\} \tag{5}$$

The functions λ_1 and λ_2 are called the dynamic eigenvalues of (1). Of course λ_1 and λ_2 depend on the chosen solution $p_1(t)$ of the Riccati equation (3), that is on the initial value $p_1(0)$ which generates $p_1(t)$. To show this, transform (4) according to

$$\left.\begin{aligned} y_1 &= y_1^{(1)} \\ y_2 &= v_1 y_1^{(1)} + y_2^{(1)} \end{aligned}\right\} \rightleftarrows \boldsymbol{y} = \mathbf{P}_2\boldsymbol{y}^{(1)} \tag{6}$$

Remark that the transformation $\mathbf{P}_1\mathbf{P}_2$ can be written as a single transformation of the same type as $\mathbf{P}_1$ and $\mathbf{P}_2$

$$\mathbf{P}_1\mathbf{P}_2 = \begin{bmatrix} 1 & \\ p_1 + v_1 & 1 \end{bmatrix} \tag{7}$$

If v_1 satisfies

$$\dot{v} = -a_{12}v^2 - (a_{11} - a_{22} + 2p_1a_{12})v \tag{8}$$

then (4) and (6) yield

$$\left.\begin{aligned} \dot{y}_1^{(1)} &= \lambda_1(t,p_1+v_1)y_1^{(1)} + a_{12}y_2^{(1)} \\ \dot{y}_2^{(1)} &= \lambda_2(t,p_1+v_1)y_2^{(1)} \end{aligned}\right\} \rightleftarrows \dot{\boldsymbol{y}}^{(1)} = \mathbf{B}_2\boldsymbol{y}^{(1)} \tag{9}$$

This suggests that $p_1 + v_1$ is also a solution of the Riccati equation (3). A simple proof confirms the assertion. Moreover it suggests that for our purpose (6) is redundant and is just a change of the initial value. Remark that (8) can easily be solved by changing the dependent variable according to

$$w = v^{-1} \tag{10}$$

so that there results a linear equation

$$\dot{w} = a_{12} + (a_{11} - a_{22} + 2p_1a_{12})w \tag{11}$$

Next (4) will be forced to a diagonal form using the second transformation

$$\left.\begin{aligned} y_1 &= z_1 + q_1z_2 \\ y_2 &= z_2 \end{aligned}\right\} \rightleftarrows \boldsymbol{y} = \mathbf{Q}_1\boldsymbol{z} \tag{12}$$

If q_1 satisfies

$$\dot{q} = \{\lambda_1(t,p_1) - \lambda_2(t,p_1)\}q + a_{12} \tag{13}$$

then

$$\left.\begin{aligned} \dot{z}_1 &= \lambda_1(t,p_1)z_1 \\ \dot{z}_2 &= \lambda_2(t,p_1)z_2 \end{aligned}\right\} \rightleftarrows \dot{\boldsymbol{z}} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \boldsymbol{z} \tag{14}$$

Apply next to (14)

$$\left.\begin{aligned} z_1 &= z_1^{(1)} + w_1 z_2^{(1)} \\ z_2 &= z_2^{(1)} \end{aligned}\right\} \rightleftarrows \boldsymbol{z} = \mathbf{Q}_2\boldsymbol{z}^{(1)} \tag{15}$$

We get (14) back, but now for $z_1^{(1)}$ and $z_2^{(1)}$, if w_1 satisfies

$$\dot{w} = \{\lambda_1(t,p_1) - \lambda_2(t,p_1)\}w \tag{16}$$

Remark that (13) and (16) yield

$$(q+w)^{\cdot} = \{\lambda_1(t,p_1) - \lambda_2(t,p_1)\}(q+w) + a_{12} \tag{17}$$

Or in operator formulation, the product operator

$$\mathbf{Q}_1\mathbf{Q}_2 = \begin{bmatrix} 1 & q_1 + w_1 \\ 0 & 1 \end{bmatrix} \tag{18}$$

is of the same type as the operators itself. So q_1 can be given any initial value to get an unique solution of (1). We can now state the solution of (1) as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{P}_q(t) \begin{bmatrix} e^{\gamma_1(t)} & 0 \\ 0 & e^{\gamma_2(t)} \end{bmatrix} \mathbf{P}_q^{-1}(0) \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \tag{19}$$

where

$$\mathbf{P}_q(t) = \begin{bmatrix} 1 & 0 \\ p(t) & 1 \end{bmatrix} \begin{bmatrix} 1 & q(t) \\ 0 & 1 \end{bmatrix} \tag{20}$$

The functions $\gamma_i(t)$ are defined by

$$\gamma_i(t) = \int_0^t \lambda_i(\tau)d\tau \tag{21}$$

III. Examples

In this section two examples will be presented to illustrate the material of section 2. Both examples concern periodic systems. The first example is a classical one ([5]).Its main purposes are to give an insight in the presented theory and to show the advantages of making choices for the initial values of $p(t)$ and $q(t)$. Moreover, it shows how periodic parts in the dynamic eigenvalues can be treated in order to obtain the Floquet decomposition. For the second example it is also the goal to derive the Floquet decomposition. Here

the matrix in the exponential of this decomposition is not a diagonal one. We show by choosing two initial values for $p(t)$ that the Floquet decomposition is not always obvious.
As first example we have

$$\begin{aligned} \dot{x}_1 &= (-1+\alpha\cos^2 t)x_1 + (1-\alpha\sin t\cos t)x_2 \\ \dot{x}_2 &= (-1-\alpha\sin t\cos t)x_1 + (-1+\alpha\sin^2 t)x_2 \end{aligned} \tag{22}$$

The system has classical eigenvalues given by

$$\det\begin{bmatrix}\lambda+1-\alpha\cos^2 t & -1+\alpha\sin t\cos t \\ 1+\alpha\sin t\cos t & \lambda+1-\alpha\sin^2 t\end{bmatrix} = 0 \leftrightarrow$$
$$\leftrightarrow \lambda_{1,2} = (\tfrac{1}{2}\alpha - 1) \pm \sqrt{\tfrac{1}{4}\alpha^2 - 1} \tag{23}$$

For $\alpha = 2$ we have $\lambda_{1,2} = 0$. For $\alpha > 2$ one eigenvalue is positive. We will show in due course that this system is unstable for $\alpha > 1$; this is not in line with the suggestions given by the eigenvalues. For $\alpha < 2$ the eigenvalues are complex conjugated with negative real part. Also here a discrepancy in use of eigenvalues. The Riccati equation reads for this example

$$\dot{p} + p^2 + 1 = \alpha[p\cos t + \sin t][p\sin t - \cos t] \tag{24}$$

This has as a solution

$$p_1 = -\tan(t) \tag{25}$$

With $p = p_1 + v_1$, (8), (10) and (11) the general solution of (24) can be obtained as

$$p(t) = \frac{p(0)\cos t - \sin t e^{\alpha t}}{p(0)\sin t + \cos t e^{\alpha t}} \tag{26}$$

Remark that for $p(0) = \infty$ the periodic solution $p_2 = \cot(t)$ is obtained. This solution together with the solution p_1 are the only periodic solutions of (24). All the others are nonperiodic. Moreover, the two solutions $p = -\tan t$ and $p = \cot t$ do not depend on the initial conditions. They serve in some way as equilibrium solutions for the Riccati equations. If $\alpha > 0$ then $p = -\tan t$ can be considered as the stable solution, while for $\alpha < 0$ the solution $p = \cot t$ is stable. We get for (5)

$$\left.\begin{aligned} \lambda_1(t,p_1) &= \alpha - 1 - \tan t \\ \lambda_2(t,p_1) &= -1 + \tan t \end{aligned}\right\} \tag{27}$$

and for (13) we get

$$\dot{q} + (2\tan t)q - 1 = \alpha[q - \sin t\cos t] \tag{28}$$

It is easy to see that a solution is

$$q_1(t) = \sin t\cos t \tag{29}$$

The general solution of (28) is

$$q(t) = \sin t\cos t + q(0)\cos^2 t e^{\alpha t} \tag{30}$$

which is not periodic unless $q(0) = 0$ is satisfied. With (21) it follows that

$$\left.\begin{aligned} \gamma_1(t) &= (\alpha-1)t + \ln|\cos t| \\ \gamma_2(t) &= -t + \ln|\cos t|^{-1} \end{aligned}\right\} \tag{31}$$

The next problem is to obtain the Floquet decomposition of (22) and the relation with (27). It is remarked that λ_i $(i = 1,2)$ is periodic, but this will not be true for γ_i $(i = 1,2)$. We state that (14) has for this example the solution

$$\left.\begin{aligned} z_1(t) &= e^{(\alpha-1)t}\cos t \\ z_2(t) &= e^{-t}(\cos t)^{-1} \end{aligned}\right\} \tag{32}$$

In (32) is already build in that we want

$$z_1(0) = z_2(0) = 1 \tag{33}$$

With (25), (29), and (31) we now obtain for (19)

$$\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}1 & \sin t\cos t \\ -\tan t & \cos^2 t\end{bmatrix}\begin{bmatrix}\cos t & 0 \\ 0 & (\cos t)^{-1}\end{bmatrix} \times$$
$$\times \begin{bmatrix}e^{(\alpha-1)t} & 0 \\ 0 & e^{-t}\end{bmatrix}\begin{bmatrix}x_1(0)\\x_2(0)\end{bmatrix} \tag{34}$$

Thus in the Floquet decomposition

$$\boldsymbol{x}(t) = \mathbf{F}(t)e^{\mathbf{S}t}\boldsymbol{x}(0) \tag{35}$$

we have

$$\mathbf{F}(t) = \begin{bmatrix}\cos t & \sin t \\ -\sin t & \cos t\end{bmatrix} \quad \mathbf{S} = \begin{bmatrix}\alpha-1 & 0 \\ 0 & -1\end{bmatrix} \tag{36}$$

The second example [5]

$$\left.\begin{aligned} \dot{x}_1 &= -x_1 \\ \dot{x}_2 &= -\cos t x_1 \end{aligned}\right\} \tag{37}$$

serves to show that the outlined procedure does not lead to a Floquet decomposition in a direct way.This depends on the chosen initial values for the Riccati equation. Secondly, this example shows that the matrix in the exponent of the Floquet decomposition is not necessarily diagonal. Here, (37) yields a simplified form of the Riccati equation

$$\dot{p} = p - \cos t \tag{38}$$

with as a general solution

$$p = \tfrac{1}{2}(\cos t - \sin t) + Ke^t \tag{39}$$

If the initial value $p(0) = 0$ is used, then we find

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{2}(\cos t - \sin t) - \frac{1}{2}e^t & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & 1 \end{bmatrix} \times$$
$$\times \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \quad (40)$$

This is not a Floquet representation. If $p(0) = \frac{1}{2}$, then we have

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{2}(\cos t - \sin t) & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & 1 \end{bmatrix} \times$$
$$\times \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \quad (41)$$

yielding a periodic part, but now the exponential part causes trouble. Using

$$\begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{bmatrix} = \exp\left\{ \begin{bmatrix} -1 & 0 \\ -\frac{1}{2} & 0 \end{bmatrix} t \right\} \quad (42)$$

it becomes obvious that (41) can be rewritten in the form (35) with

$$\mathbf{F}(t) = \begin{bmatrix} 1 & 0 \\ \frac{1}{2}(\cos t - \sin t) - \frac{1}{2} & 1 \end{bmatrix}, \mathbf{S} = \begin{bmatrix} -1 & 0 \\ -\frac{1}{2} & 0 \end{bmatrix} \quad (43)$$

Note that $\mathbf{S}$ is not diagonal.

IV. The Relation Between Floquet Numbers Ans Dynamic Eigenvalues

In the preceding section, we have shown for two examples, (22) and (37), that their modal solutions can be transformed into their Floquet representations (35), (36) and (43). Both examples show that the mean value of the dynamic eigenvalues equals to the Floquet numbers of the differential equation.
To demonstrate this in a more general sense, the modal solution (19) of (1) will be written as

$$\boldsymbol{x}(t) = \mathbf{P}_q(t) \begin{bmatrix} e^{\gamma_1(t)} & 0 \\ 0 & e^{\gamma_2(t)} \end{bmatrix} \mathbf{P}_q^{-1}(0)\boldsymbol{x}(0) \quad (44)$$

The solution $p(t)$ in $\mathbf{P}_q(t)$ is assumed to be a periodic solution of (3). In [6] is indicated when there are periodic solutions $p(t)$. Then also the dynamic eigenvalues $\lambda_1(t)$ and $\lambda_2(t)$ are periodic. With the theory of Fourier series, their integrals $\gamma_1(t)$ and $\gamma_2(t)$ thus have a linear component and a periodic part. Let us write

$$\gamma_i(t) = \{\gamma_i(t) - \bar{\lambda}_i t\} + \{\bar{\lambda}_i t\} \quad (45)$$

where

$$\bar{\lambda}_i = \frac{1}{T} \int_0^T \lambda_i(\tau) d\tau \quad (46)$$

So that

$$\begin{bmatrix} e^{\gamma_1(t)} & 0 \\ 0 & e^{\gamma_2(t)} \end{bmatrix} = \begin{bmatrix} e^{\gamma_1(t) - \bar{\lambda}_1 t} & 0 \\ 0 & e^{\gamma_2(t) - \bar{\lambda}_2 t} \end{bmatrix} \begin{bmatrix} e^{\bar{\lambda}_1 t} & 0 \\ 0 & e^{\bar{\lambda}_2 t} \end{bmatrix} \quad (47)$$

And (19) can be written as (35) with

$$\mathbf{F}(t) = \mathbf{P}_q(t) \begin{bmatrix} e^{\gamma_1(t) - \bar{\lambda}_1 t} & 0 \\ 0 & e^{\gamma_2(t) - \bar{\lambda}_2 t} \end{bmatrix} \mathbf{P}_q^{-1}(0) \quad (48)$$

and

$$\mathbf{S} = \mathbf{P}_q(0) \begin{bmatrix} \bar{\lambda}_1 & 0 \\ 0 & \bar{\lambda}_2 \end{bmatrix} \mathbf{P}_q^{-1}(0) \quad (49)$$

V. Conclusions

In this paper the Floquet representation for the solution of a periodic differential equation is derived. First the modal solution is obtained which is the sum of number of modes. Each mode is the product of a dynamic eigenvector and an exponential whose argument is the integral of a dynamic eigenvalue.
It is argued that under certain conditions the dynamic eigenvalues are periodic and that the exponentials are the product of an exponential with a periodic argument and a second exponential with a linear argument. It is remarked that the slope of the linear arguments equals the mean value of the dynamic eigenvectors over one period of the coefficients, so this mean value is a Floquet number. Since the dynamic eigenvector is obtained solving a differential equation of Riccati, it is not necessary first to solve the original differential equation in order to obtain the Floquet numbers!

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