SCHEMES OF POLYNOMIAL EQUATIONS THAT CHARACTERIZE THE VARIATIONAL BEHAVIOUR

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Abstract— *The variational equations of nonlinear dynamical systems are addressed. It is shown that their dynamics is fully characterized by a set of polynomials of decreasing order.*

I. Introduction

As is well known, small variations around a solution trajectory of general dynamical systems satisfy a linear time-varying (LTV) equation [1]. On their turn, the modal solutions of LTV equations are fully characterized by the earlier introduced dynamical eigenvalues [2].

In this paper, it is shown that the dynamical eigenvalues satisfy a scheme of polynomial equations of decreasing order. The time-dependent coefficients of each polynomial equations incorperate a dynamical eigenvalue solution of a lower order polynomial.

At first glance this seems to be in contradiction with the theory of linear time-invariant (LTI) systems with precisely one characteristic equation for the complete eigenspectrum.

However, in Section 2 it will be shown that also for LTI systems there is a set of characteristic equations corresponding to the set of eigenvalues.

It is derived with respect to single-input single-output (SISO) systems. In Section 3, the state space approach for time-varying systems is used in order to obtain the dynamic eigenvalues and the corresponding characteristic equations. The results are in agreement with LTI systems. It is demonstrated that the set of characteristic equations for LTV systems cannot be reduced to one and the same characteristic equation. As a consequence, our results are generalizations of the work of Kamen [3] and Zhu [4], respectively.

Finally, it is shown how the Cauchy-Floquet decomposition can be obtained without using the state space approach.

II. Scheme Of Characteristic Equations For Lti-Systems

Assume that the homogeneous input-output relation for SISO-LTI-systems is given by

$$a_0 D^n x + a_1 D^{n-1} x + \cdots + a_{n-1} D x + a_n x = 0\,, \quad (1)$$

where $D = d/dt$ and $a_0, a_1, \ldots, a_n$ are constant coefficients, respectively. Relation (1) will be normalized by setting

$$a_0 = 1\,. \quad (2)$$

In (1) we write, for each time derivative

$$D^k x = D^{k-1}[D - \lambda]x + \lambda D^{k-1} x\,. \quad (3)$$

Then, the input-output equation (1) can be rewritten as

$$\sum_{i=0}^{n-1} \alpha_i D^{n-1-i}[D-\lambda]x + \left(\sum_{i=0}^{n} a_i \lambda^{n-i}\right) x = 0\,, \quad (4)$$

in which

$$\alpha_i = \sum_{j=0}^{i} a_j \lambda^{i-j} \quad (i = 0, 1, \ldots, n-1)\,. \quad (5)$$

Equation (4) shows that a modal solution of the form

$$x = \exp(\lambda t) \quad (6)$$

satisfies (1) if and only if eigenvalue λ is a solution of the polynomial equation

$$\sum_{i=0}^{n} a_i \lambda^{n-i} = 0\,, \quad (7)$$

which at this place is called the *first characteristic equation*. Note that in view of (7) it follows that $\alpha_n = 0$. In a further expansion, equation (4) can be written as

$$\sum_{i=0}^{n-2} \beta_i D^{n-2-i}[D-\mu][D-\lambda]x+$$
$$+\left(\sum_{i=0}^{n-1} \alpha_i \mu^{n-1-i}\right)[D-\lambda]x+\left(\sum_{i=0}^{n} a_i \lambda^{n-i}\right) x = 0 \tag{8}$$

in which

$$\beta_i = \sum_{j=0}^{i} \alpha_j \mu^{i-j} \quad (i = 0, 1, \ldots, n-1)\,. \tag{9}$$

Now, equation (8) shows that the modal solution

$$x = \exp(\mu t) \tag{10}$$

satisfies the LTI input-output equation (1) if and only if

$$\left(\sum_{i=0}^{n-1} \alpha_i \mu^{n-1-i}\right)[\mu-\lambda] + \sum_{i=0}^{n} a_i \lambda^{n-i} = 0\,. \tag{11}$$

Next, if (6) is a solution of (1), equation (11) reduces in view of (7) to a so-called *second characteristic equation*

$$\sum_{i=0}^{n-1} \alpha_i \mu^{n-1-i} = 0\,. \tag{12}$$

Moreover, the solution (11) yields by substituting of (5) for α_i

$$\left(\sum_{i=0}^{n-1} \alpha_i \mu^{n-1-i}\right)[\mu-\lambda] + \sum_{i=0}^{n} a_i \lambda^{n-i} =$$
$$= \sum_{i=0}^{n} a_i \lambda^{n-i} - \sum_{i=0}^{n-1} a_j \lambda^{n-j} + \left(\sum_{j=0}^{0} a_j \lambda^{0-j}\right)\mu^n +$$
$$+\left(\sum_{j=0}^{1} a_j \lambda^{1-j} - \sum_{j=0}^{0} a_j \lambda^{1-j}\right)\mu^{n-1} + \ldots$$
$$\cdots + \left(\sum_{j=0}^{n-1} a_j \lambda^{n-1-j} - \sum_{j=0}^{n-1} a_j \lambda^{n-1-j}\right)\mu^1 =$$
$$a_0\mu^n + a_1\mu^{n-1} + \cdots + a_{n-1}\mu^1 + a_n\,. \tag{13}$$

Thus, for LTI systems the second characteristic equation is observed to be equivalent to the first characteristic equation.
From another point of view, equation (7) gives an algebraic polynomial of degree n, while (12) yields an algebraic polynomial of degree $n-1$.
Furthermore, relation (5) directly implies for ($i = 1, 2, \ldots, n-1$)

$$\lambda\alpha_{n-i} = -a_{n-i+1} + \alpha_{n-i+1} \tag{14}$$

and for $i = 0$

$$a_0\lambda = -a_1 + \alpha_1 \quad \text{with} \quad a_0 = 1\,. \tag{15}$$

Now, if λ is eliminated from (14) and (15), we obtain for $i = 1, 2, \ldots, n-1$

$$(\alpha_1 - a_1)\alpha_{n-i} - \alpha_{n-i+1} + a_{n-i+1} = 0\,. \tag{16}$$

Next, we introduce the row vectors

$$\left.\begin{aligned} \boldsymbol{\alpha}^T &= [\,\alpha_{n-1}, \ldots, \alpha_1\,] \\ \boldsymbol{a}^{\boldsymbol{T}} &= [\,a_n, \ldots, a_2\,] \\ \boldsymbol{e}_{\boldsymbol{n-1}}^{\boldsymbol{T}} &= [\,0, \ldots, 0, 1\,] \end{aligned}\right\} \tag{17}$$

in which T stands for the transpose and the shift-matrix

$$\mathbf{I}_{n-1}^{+} = \begin{bmatrix} 0 & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \ldots & \ldots & 0 \end{bmatrix}\,. \tag{18}$$

As a consequence, (16) can be written as the vector algebraic Riccati equation [5]

$$\boldsymbol{\alpha}^T \boldsymbol{e}_{n-1} \boldsymbol{\alpha}^T - a_1 \boldsymbol{\alpha}^T - \boldsymbol{\alpha}^T \mathbf{I}_{n-1}^{+} + \boldsymbol{a}^{\boldsymbol{T}} = \boldsymbol{0}^{\boldsymbol{T}}\,. \tag{19}$$

In the same way, the characteristic equation (12) together with (9) induces a second algebraic Riccati equation, namely

$$\boldsymbol{\beta}^T \boldsymbol{e}_{n-2} \boldsymbol{\beta}^T - b_1 \boldsymbol{\beta}^T - \boldsymbol{\beta}^T \mathbf{I}_{n-2}^{+} + \boldsymbol{b}^{\boldsymbol{T}} = \boldsymbol{0}^{\boldsymbol{T}} \tag{20}$$

with

$$\boldsymbol{\beta}^T = [\,\beta_{n-2}, \ldots, \beta_1\,], \boldsymbol{b}^{\boldsymbol{T}} = [\,\alpha_{\boldsymbol{n-1}}, \ldots, \alpha_{\boldsymbol{2}}\,], \boldsymbol{b}_{\boldsymbol{1}} = \alpha_{\boldsymbol{1}}\,. \tag{21}$$

This process can be continued $(n-1)$ times. The final result is that the original input-output equation (1) is replaced by

$$[D-\lambda_1][\ldots][D-\lambda_n]x + \sum_{i=0}^{1} \alpha_i^{(n-1)} \lambda_1^{1-i}[D-\lambda_2][\ldots]$$
$$\times[D-\lambda_n]x + \sum_{i=0}^{2} \alpha_i^{(n-2)} \lambda_2^{2-i}[D-\lambda_3][\ldots][D-\lambda_n]x +$$
$$\cdots + \sum_{i=0}^{n-1} \alpha_i^{(1)} \lambda_{n-1}^{n-1-i}[D-\lambda_n]x + \sum_{i=0}^{n} \alpha_i^{(0)} \lambda_n^{n-i} x = 0. \tag{22}$$

It is concluded that the original differential polynomial with constant coefficients in (1) is *factorized from the right* with the eigenvalues $\lambda_n, \lambda_{n-1}, \dots, \lambda_1$. The coefficients $\alpha_i^{(j)}$ $(j = 0, 1, \dots, n-1)$ are obtained as

$$\alpha_i^{(0)} = a_i \; (i = 0, 1, \dots, n) , \tag{23}$$

$$\alpha_i^{(j)} = \sum_{k=0}^{i} \alpha_k^{(j-1)} \lambda_j^{n+1-j} \text{for} \begin{cases} (i = 0, 1, \dots, n-j) \\ (j = 1, 2, \dots, n-1) , \end{cases} \tag{24}$$

with

$$\alpha_0^{(j)} = 1 . \tag{25}$$

The elimination of the eigenvalues λ_j from (24) with $i = 1$ leads to

$$\alpha_1^{(j)} = \alpha_1^{(j-1)} + \lambda_j \tag{26}$$

which on its turn yields on account of (25) a set algebraic Riccati equations a for lower dimension.

III. Scheme Of Characteristic Equations For Ltv-Systems

In the preceding section a scheme of characteristic equations for a LTI system has been derived. In it, each equation corresponds to a single algebraic Riccati equation. In this section the reverse problem will be considered: the Riccati equation will be obtained directly from the differential equation, and afterwards the characteristic equation from the Riccati equation (compare [6]). For that purpose, the input-output equation (1) is rewritten in the state space description

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \mathbf{I}_{n-1}^{+} & \boldsymbol{e}_{n-1} \\ -\boldsymbol{a}^T & -a_1 \end{bmatrix} \boldsymbol{x} , \tag{27}$$

where the dot stands for a differentiation with respect to the time t. This equation will be transformed to a second state space description according to the transformation

$$\boldsymbol{x} = \begin{bmatrix} \mathbf{I}_{n-1} & \boldsymbol{0} \\ \boldsymbol{p}^T & 1 \end{bmatrix} \boldsymbol{y} , \tag{28}$$

in which

$$\boldsymbol{p}^T = [p_1, \dots, p_{n-1}] . \tag{29}$$

The result of this transformation can be stated as

$$\dot{\boldsymbol{y}} = \begin{bmatrix} \mathbf{I}_{n-1}^{+} + \boldsymbol{e}_{n-1}\boldsymbol{p}^T & \boldsymbol{e}_{n-1} \\ \boldsymbol{0}^{\boldsymbol{T}} & \tilde{\lambda}_n \end{bmatrix} \boldsymbol{y} , \tag{30}$$

where

$$\tilde{\lambda}_n = -a_1 - p_{n-1} \tag{31}$$

and $\boldsymbol{p}^{\boldsymbol{T}}$ satisfies the vector Riccati differential equation

$$\dot{\boldsymbol{p}}^T = -\boldsymbol{p}^T \mathbf{I}_{n-1}^{+} - \boldsymbol{a}^T + \tilde{\lambda}_n \boldsymbol{p}^T . \tag{32}$$

It may be clear that in (28) we have assumed that the vector $\boldsymbol{p}$ is a function of time, thus $\boldsymbol{p}^T = \boldsymbol{p}^T(t)$. This allows a generalization to LTV systems.
If $\boldsymbol{p}^{\boldsymbol{T}}$ is assumed to be a constant, then the left-hand side of (32) reduces to zero and we have an algebraic Riccati equation. In that case, (30) shows that $\tilde{\lambda}_n$ is an classical eigenvalue of system given by (30) and thus of system (27). As a consequence, $\tilde{\lambda}_n$ is an eigenvalue of the original system, given by (1).
Next, we have to show that $\tilde{\lambda}_n = \tilde{\lambda}_n(t)$ satisfies a characteristic equation. For that purpose (32) is rewritten as

$$-p_i + \tilde{\lambda}_n p_{i+1} = a_{n-i} + \dot{p}_{i+1} \; (i = 0, 1, \dots, n-2) \tag{33}$$

with $p_0 = 0$. If the equations in (33) are multiplied by $\tilde{\lambda}_n^i$ and subsequently added together, we obtain

$$\tilde{\lambda}_n^{n-1} p_{n-1} = \sum_{i=2}^{n} (a_i + \dot{p}_{n-i+1}) \tilde{\lambda}_n^{n-i} . \tag{34}$$

Elimination of p_{n-1} from (33) with the aid of (31) yields

$$\sum_{i=1}^{n} \bar{a}_i \tilde{\lambda}_n^{n-i} = 0 , \tag{35}$$

in which the modified polynomial time-dependent coefficients $\bar{a}_i = \bar{a}_i(t)$ are given by

$$\bar{a}_i = a_i + \dot{p}_{n-i+1} , \tag{36}$$

with $p_n = 0$. Thus for LTI systems, where $\boldsymbol{p}^{\boldsymbol{T}}$ is a constant vector ($\dot{p}_{n-i+1} = 0$), equation (35) equals indeed the classical characteristic equation with $\tilde{\lambda}_n$ an eigenvalue of the input-output equation (1).
Next, we show

$$\tilde{\lambda}_n = \lambda_1 . \tag{37}$$

To that aim, we remark that (30) yields

$$\left. \begin{aligned} \dot{y}_i &= y_{i+1} \quad (i = 1, 2, \dots, n-2) \\ \dot{y}_{n-1} &= p_1 y_1 + \dots + p_{n-1} y_{n-1} + y_n \\ \dot{y}_n &= \tilde{\lambda}_n y_n \end{aligned} \right\} . \tag{38}$$

As a consequence, we have

$$\left.\begin{array}{r} D^{n-1}y_1 - p_{n-1}D^{n-2}y_1 - \dots \\ \dots - p_2Dy_1 - p_1y_1 = y_n \\ \dot{y}_n = \tilde{\lambda}_n y_n \end{array}\right\} . \quad (39)$$

Hence, y_n has the modal form [7]

$$y_n(t) = C \exp[\int^t \tilde{\lambda}(\tau) d\tau] , \quad (40)$$

with C a constant. In addition, we have

$$[D - \tilde{\lambda}_n](D^{n-1} - p_{n-1}D^{n-2} - \dots - p_2D - p_1)y_1 = 0. \quad (41)$$

It is observed that the original differential polynomial in (1) will be *factorized from the left* this time. Since the transformation (28) implies

$$y_1 = x_1 = x , \quad (42)$$

equation (41) directly results into the identity (37).

It should be remarked, again, that (41) remains valid if the coefficients a_i are functions of time. To show this directly without the use of any state space description, write

$$D^n x = [D - \tilde{\lambda}_n]D^{n-1}x + \tilde{\lambda}_n D^{n-1}x \quad (43)$$

and substitute (31), resulting into

$$D^n x = [D - \tilde{\lambda}_n]D^{n-1}x - (a_1 + p_{n-1})D^{n-1}x. \quad (44)$$

As a consequence, we obtain

$$D^n x + a_1 D^{n-1}x = [D - \tilde{\lambda}_n]D^{n-1}x - p_{n-1}D^{n-1}x . \quad (45)$$

Next in (45) we apply

$$p_{n-1}D^{n-1}x = D(p_{n-1}D^{n-2}x) - \dot{p}_{n-1}D^{n-2}x = [D - \tilde{\lambda}_n]p_{n-1}D^{n-2}x + (\tilde{\lambda}_n p_{n-1} - \dot{p}_{n-2})D^{n-2}x \quad (46)$$

and subsequently use the expression (33) for $i = n - 2$ This yields

$$D^n x + a_1 D^{n-1}x + a_2 D^{n-2}x = [D - \tilde{\lambda}_n](D^{n-1}x - p_{n-1}D^{n-2}x) - p_{n-2}D^{n-2}x . \quad (47)$$

By repetition of the above arguments we get

$$D^n x + a_1 D^{n-1}x + a_2 D^{n-2}x + \dots + a_{n-1}Dx = [D - \tilde{\lambda}_n](D^{n-1}x - p_{n-1}D^{n-2}x - \dots - p_2Dx) - p_1Dx. \quad (48)$$

Finally, with

$$p_1 Dx = [D - \tilde{\lambda}_n]p_1 x + a_n x , \quad (49)$$

we arrive at

$$D^n x + a_1 D^{n-1}x + a_2 D^{n-2}x + \dots + a_{n-1}Dx + a_n x = [D - \tilde{\lambda}_n](D^{n-1}x - p_{n-1}D^{n-2}x - \dots - p_2Dx - p_1x). \quad (50)$$

It is clear that this process can be continued until the Cauchy-Floquet decomposition is obtained.

IV. Conclusions

In this paper, it is argued that for linear time-invariant (LTI) as well as for time-varying (LTV) systems each term of the Cauchy-Floquet factorization of the differential operator induces a characteristic polynomial and a set of coupled Riccati equations. For a n-th order system, the first right placed factor gives a n-th order characteristic polynomial and $n-1$ coupled algebraic Riccati equations, with $n-1$ solutions. For constant systems, these $n-1$ solutions are the coefficients in the remaining differential polynomial. For LTV-systems, the vector algebraic Riccati equation is replaced by a vector differential Riccati equation, resulting in modified time dependent coefficients of the polynomials.

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