Lyapunov Exponents and Dynamic Eigenvalues

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Abstract— **An unification of different definitions of the Lyapunov exponent is given. This is done by showing that Lyapunov exponents are the limit of the ergodic mean values of the earlier introduced dynamic eigenvalues.**

I. Introduction

In the literature, there are a number of definitions in use for the notion of a Lyapunov exponent. First, one can distinguish between continuous dynamical systems (as for example electrical circuits and mechanical systems) and discrete dynamical systems (iteration schemes and so on) [1]. Secondly, one can distinguish between the case based on the solution of a differential equation and the case based on the flow associated with the differential equation [2].
Moreover, in a number of recent papers [3], [4], [5], [6] the concepts of dynamic eigenvalues and dynamic eigenvectors were introduced. They describe the dynamic behavior of a linear time-varying differential system. As a consequence, these new concepts should also be valuable when the asymptotic behavior of the solution is discussed. And, by definition, this is also done by the Lyapunov exponent. Thus Lyapunov exponents must be related to dynamic eigenvalues.
In [8], it is sketched how the relation is between the Lyapunov exponents and the singular values of the fundamental matrix associated with a linear time-varying system. As a consequence, the Lyapunov exponents are real numbers. For linear time-invariant systems this is a sound approach.
For a linear time-varying system however, a problem arises. To sketch this problem, we first consider the ellipsoid associated with the fundamental matrix of the system. In a constant system this ellipsoid is invariant with respect to the chosen frame of reference by which the system is described. Secondly, we mention that changes in the frame of reference are described by similarity transformations in the invariant case. This no longer holds for time-varying systems. As will be shown for time-varying systems, a change in frame of reference asks for a Riccati transform in stead of a similarity transformation [7]. Only for a frame of reference in which the system equations are decoupled, the information of invariant systems can be used for time-varying systems.
So, our first definition of the Lyapunov exponent will be a slight extension of the definition given in [8].
Since the fundamental solution contains all the information for the solution of the system of differential equations, one can also write down the shape of the solution for each state variable if the fundamental solution is known. We will analyse such a solution in order to substract the information of a Lyapunov exponent. This will yield a formulae which also can be used as a definition for the Lyapunov exponent.
With the aid of the fundamental matrix, one can introduce the flow associated with the differential equation. With this flow an iterated mapping can be defined and a third definition for the Lyapunov exponent will be introduced [10]. This definition is given in discrete time, so the usual problems with dimensions and units will occur when converting from continuous time to discrete time. This will be shown for a scalar system.
The three different definitions for the Lyapunov exponent and their relation to the dynamic eigenvalues will be presented in Section 3 (the first and second definition) and Section 4 (the third definition), respectively.
In Section 2 it is shown how the modal solution of a homogeneous linear time-varying system of differential equations is derived.
This process uses a product of transformations, each of them being a Riccati transformation. Each of them generates a characteristic equation for the homogeneous linear time-varying system. This characteristic equation reduces to the well-known expression for invariant systems. This is shown in the accompanying paper [11].
The product of transformations yields a triangular system matrix. For this particular system matrix analytic solutions can be deduced and, as a consequence, also the solution for the original differential system can be obtained.
In Section 5 finally, some conclusions will be formulated.

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II. Modal solutions

Consider the differential equation

$$\dot{\boldsymbol{x}}(t) = \mathbf{A}(t)\boldsymbol{x}(t)\,, \tag{1}$$

with $\boldsymbol{x}(t) \in \mathbb{R}^n$ and $\mathbf{A}(t) \in \mathbb{R}^{n\times n}$. The existence of a Lipschitz condition is assumed in order to achieve that (1) has a unique solution for $t \geq t_0$ depending on the initial condition

$$\boldsymbol{x}_0 = \boldsymbol{x}(t_0)\,. \tag{2}$$

There exists a compound state transformation

$$\boldsymbol{x}(t) = \mathbf{R}(t)\boldsymbol{y}(t)\,, \tag{3}$$

with

$$\mathbf{R} = \mathbf{R}^{(n)}\mathbf{R}^{(n-1)}\ldots\mathbf{R}^{(2)} \quad \text{and} \quad \forall\mathbf{R}^{(k)} \in \mathbb{R}^{n\times n}\,, \tag{4}$$

where the superscript within parenthesis reflects the iteration steps.
On account of transformation (3), together with (4) we obtain from (1)

$$\dot{\boldsymbol{y}}(t) = \mathbf{B}(t)\boldsymbol{y}(t)\,, \tag{5}$$

with $\mathbf{B}(t) \in \mathbb{R}^{n\times n}$ an upper triangle matrix.
The transformation process can be described by the following iteration scheme [12]

$$\mathbf{A}^{(n)} = \mathbf{A} \tag{6}$$

$$\mathbf{A}^{(k-1)} = \left[\mathbf{R}^{-1}\mathbf{A}\mathbf{R} - \mathbf{R}^{-1}\dot{\mathbf{R}}\right]^{(k)} \;(k = n, \ldots, 2)\,, \tag{7}$$

where $[\quad]^{(k)}$ denotes that all matrices within the brackets have superscript k. This process finally yields

$$\mathbf{B} = \mathbf{A}^{(1)}\,, \tag{8}$$

where $\mathbf{B}$ is the triangular matrix in (5).
The matrix $\mathbf{R}^{(n-k)}$ $(k = 0, 1, \ldots, n-2)$ is given by

$$\mathbf{R}^{(n-k)} = \begin{bmatrix} \mathbf{P}_{n-k} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_k \end{bmatrix}, \tag{9}$$

where $\mathbf{P}_{n-k} \in \mathbb{R}^{(n-k)\times(n-k)}$ and $\mathbf{I}_k \in \mathbb{R}^{k\times k}$ the unit matrix, respectively.
Moreover

$$\mathbf{P}_{n-k} = \begin{bmatrix} 1 & \cdots & \cdots & 0 \\ \vdots & \ddots & \mathbf{0} & \vdots \\ \vdots & \mathbf{0} & \ddots & \vdots \\ p_{n-k,1} & \cdots & p_{n-k,n-k-1} & 1 \end{bmatrix} \tag{10}$$

where the functions $p_{n-k,1}, \ldots, p_{n-k,n-k-1}$ has to satisfy Riccati differential equations [4], [5]. The accompanying paper [11] shows how this set of differential equations is related to a set of characteristic equations.
The matrix $\mathbf{B}(t)$ in (5) is specified by

$$\mathbf{B}(t) = \begin{bmatrix} \lambda_1 & \cdots & \cdots & \vdots \\ \vdots & \ddots & b_{ij} & \vdots \\ \vdots & \mathbf{0} & \ddots & \vdots \\ \vdots & \cdots & \cdots & \lambda_n \end{bmatrix}. \tag{11}$$

The asumption that

$$\boldsymbol{y}(t) = \mathbf{F}(t)\boldsymbol{\Gamma}(t)\boldsymbol{y}(0)\,, \tag{12}$$

with

$$\mathbf{F}(t) = \begin{bmatrix} 1 & \cdots & \cdots & \vdots \\ \vdots & \ddots & f_{ij}(t) & \vdots \\ \vdots & \mathbf{0} & \ddots & \vdots \\ \vdots & \cdots & \cdots & 1 \end{bmatrix} \tag{13}$$

and

$$\boldsymbol{\Gamma} = \mathrm{Diag}\{\int_0^t \lambda_i(\tau)d\tau\} = \mathrm{Diag}\{\gamma_i(t)\}\,, \tag{14}$$

shows that $\mathbf{F}(t)$ has to satisfy

$$\dot{\mathbf{F}} + \mathbf{F}\boldsymbol{\Gamma} = \mathbf{B}\mathbf{F}\,. \tag{15}$$

Equation (15) has been studied by Gantmacher [13]. The existence of solutions is guaranteed if $\boldsymbol{\Lambda}$ and $\mathbf{B}$ have the same algebraic eigenvalues. On account of the particular structure of $\boldsymbol{\Lambda}$ and $\mathbf{B}$, it is easily observed that this condition is satisfied, indeed.
We have with (11) and (13)

$$\sum_{k=i}^{n} b_{ik}(t)f_{kj}(t) = 0 \quad (i > j) \tag{16}$$

and

$$\sum_{k=i}^{n} b_{ik}(t)f_{kj}(t) = \lambda_i(t) \quad (i = j)\,. \tag{17}$$

As a consequence, (15) can be read for $i < j$ as a set of differential equations

$$\dot{f}_{ij} + f_{ij}\lambda_j = \lambda_i f_{ij} + \sum_{k=1}^{j-i-1} b_{i,i+k}f_{i+k,j} + b_{ij}\,. \tag{18}$$

Thus, first $f_{12}, f_{23}, f_{34}, \ldots, f_{n-1,n}$ have to be solved, then $f_{13}, f_{24}, f_{35}, \ldots, f_{n-2,n}$ and finally f_{1n}. As a consequence, the solution of (1) is given by

$$\boldsymbol{x}(t) = \boldsymbol{\Phi}(t)\boldsymbol{x}(0)\,, \tag{19}$$

or

$$\boldsymbol{x}(t) = \boldsymbol{\Phi}(t, t_0)\boldsymbol{x}(t_0)\,. \tag{20}$$

In (19) and (20) we have

$$\boldsymbol{\Phi}(t) = \mathbf{R}(t)\mathbf{F}(t)\mathrm{Diag}[\exp\{\gamma_k(t)\}]\,, \tag{21}$$

$$\boldsymbol{\Phi}(t, t_0) = \boldsymbol{\Phi}(t)\boldsymbol{\Phi}^{-1}(t_0) \tag{22}$$

and

$$\gamma_k(t) = \int_0^t \lambda_k(\tau)d\tau\,. \tag{23}$$

Moreover, it is mentioned that each component $x_i(t)$ in $\boldsymbol{x}(t)$ can be written as

$$x_i(t) = r_{ij}(t)f_{jk}(t)\exp\{\gamma_k(t)\}x_k(0)\,. \tag{24}$$

The righthand side has to be summed over the indices j and k (Ricci notation).

III. Definitions for lyapunov exponents

It is clear from (3) and (12) that

$$\boldsymbol{x}(t) = \mathbf{R}(t)\mathbf{F}(t)\boldsymbol{z}(t) \tag{25}$$

defines a Riccati-Lyapunov transformation by which (1) is transformed into

$$\dot{\boldsymbol{z}}(\boldsymbol{t}) = \boldsymbol{\Lambda}(\boldsymbol{t})\boldsymbol{z}(\boldsymbol{t})\,, \tag{26}$$

where

$$\boldsymbol{\Lambda}(t) = \mathrm{Diag}\{\lambda_i(t)\}\,. \tag{27}$$

The differential equation (26) defines a linear flow given by

$$\boldsymbol{z}(t) = \boldsymbol{\Phi}_{\boldsymbol{\Lambda}}(t)\boldsymbol{z}(0)\,, \tag{28}$$

with

$$\boldsymbol{\Phi}_{\boldsymbol{\Lambda}}(t) = \mathrm{Diag}\{\exp[\gamma_i(t)]\}\,. \tag{29}$$

This flow maps a point from the n-dimensional unit ball into a point of an ellipsoid. The principal axes of this ellipsoid are given by the singular values $\sigma_1(t),\ldots,\sigma_n(t)$ of $\boldsymbol{\Phi}_{\boldsymbol{\Lambda}}(t)$. Their square roots are the eigenvalues of $\boldsymbol{\Phi}^{\mathrm{H}}_{\boldsymbol{\Lambda}}(t)\boldsymbol{\Phi}_{\boldsymbol{\Lambda}}(t)$, where $\boldsymbol{\Phi}^{\mathrm{H}}_{\boldsymbol{\Lambda}}(t)$ is the conjugate transpose of $\boldsymbol{\Phi}_{\boldsymbol{\Lambda}}(t)$.
Thus it follows that

$$\sigma_i^2(t) = \exp[\gamma_i(t)]^* \exp[\gamma_i(t)]\,, \tag{30}$$

where $*$ denotes the complex conjugate,

$$\sigma_i^2(t) = \exp[2\,\mathrm{Re}\,\gamma_i(t)]\,. \tag{31}$$

As a consequence, by (23) and (31) this yields

$$\mathrm{Re}[t^{-1}\int_0^t \lambda_i(\tau)d\tau] = t^{-1}\ln\sigma_i(t)\,. \tag{32}$$

According to [9] the Lyapunov exponent χ_i is defined as

$$\chi_i = \lim_{t\to\infty} t^{-1}\ln\sigma_i(t)\,. \tag{33}$$

This generates that the Lyapunov exponent is to be interpretated as an average of the real part of a dynamic eigenvalue integrated over a sufficient long time period.
Now it is clear that we are in a position to define a complex Lyapunov exponent by (compare [14])

$$L_i = \lim_{t\to\infty} t^{-1}\int_0^t \lambda_i(\tau)d\tau\,, \tag{34}$$

or with (23)

$$L_i = \lim_{t\to\infty} t^{-1}\gamma_i(t)\,, \tag{35}$$

thus

$$\chi_i = \mathrm{Re}(L_i)\,. \tag{36}$$

Further, the solution of (1) can be written as

$$\boldsymbol{x}(t) = \boldsymbol{\Phi}(t)\boldsymbol{x}(0)\,. \tag{37}$$

With (25) and (28) we see

$$\boldsymbol{\Phi}(t) = \mathbf{R}(t)\mathbf{F}(t)\boldsymbol{\Phi}_{\boldsymbol{\Lambda}}(t)\mathbf{F}^{-1}(0)\mathbf{R}^{-1}(0)\,. \tag{38}$$

As a consequence $\boldsymbol{\Phi}(t)$ and $\boldsymbol{\Phi}_{\boldsymbol{\Lambda}}(t)$ are not similar matrices. Although they have the same dynamical eigenvalues, they do not have the same classical algebraic eigenvalues neither the same singular values. This can be clarified by writing (38) as

$$\begin{aligned}\boldsymbol{\Phi}(t) =&[\mathbf{R}(t)\mathbf{F}(t)\boldsymbol{\Phi}_{\boldsymbol{\Lambda}}(t)\mathbf{F}^{-1}(t)\mathbf{R}^{-1}(t)]\\ &[\mathbf{R}(t)\mathbf{F}(t)\mathbf{F}^{-1}(0)\mathbf{R}^{-1}(0)]\,.\end{aligned} \tag{39}$$

Thus if all elements of $\mathbf{R}(t)$ and $\mathbf{F}(t)$ are rather flat, then $\boldsymbol{\Phi}(t)$ and $\boldsymbol{\Phi}_{\boldsymbol{\Lambda}}(t)$ are approximately similar. This condition is the same as the condition that the Riccati-Lyapunov transformation nearly equals a similarity transformation.
Since the asymptotic properties of $\boldsymbol{x}(t)$ and $\boldsymbol{z}(t)$ should be the same, it is reasonable to start with (34) as the definition for the complex Lyapunov exponent.
With (34) as definition of the complex Lyapunov exponent, we deal with a second definition frequently used in literature. To that aim, we write

$$\begin{aligned}&\exp[\gamma_k(t)] =\\ &\exp[\gamma_k(t) - L_k t]\exp[(L_k - L_1)t]\exp[L_1 t]\,.\end{aligned} \tag{40}$$

If

$$\chi_1 > \chi_k \quad (k = 2,3,\ldots,n)\,, \tag{41}$$

then we obtain

$$\lim_{t\to\infty}\exp[(L_k - L_1)t] = 0\,. \tag{42}$$

Moreover, we have

$$\lim_{t\to\infty}\exp[\gamma_k(t) - L_k t] = 1\,. \tag{43}$$

Thus each of the components $x_i(t)$ of $\boldsymbol{x}(t)$ has an exponential function $\exp[L_1 t]$ and an amplitude function $A_i(t)$ such that

$$x_i(t) \to A_i(t)\exp[L_1 t] \text{ if } t\to\infty \;\; (i = 1,2,\ldots,n)\,. \tag{44}$$

One now directly concludes from (44)

$$\lim_{t\to\infty} t^{-1}|x_i(t)| = \lim_{t\to\infty} t^{-1}|A_i(t)| + \chi_1\,. \tag{45}$$

Thus if the amplitude function has a regular behavior, then (45) yields

$$\chi_i = \mathrm{Re}\,L_i = \lim_{t\to\infty} t^{-1}|x_i(t)|\,. \tag{46}$$

IV. The third definition

To make a connection with the Lyapunov exponent as used in time-discrete or iteration procedures, we consider the first order scalar system

$$\dot{x}(t) = a(t)x(t)\,. \tag{47}$$

This has as solution

$$x(t) = \exp[\int_0^t a(\tau)d\tau]x(0)\,. \tag{48}$$

This introduces the recurrent relation

$$x_{n+1} = f(x_n)\,, \tag{49}$$

where f is a linear mapping

$$f(x_n) = \exp[\int_{t_n}^{t_{n+1}} a(\tau)d\tau]x_n\,. \tag{50}$$

Here

$$x_n = x(t_n) \quad (n = 0, 1, \ldots) \tag{51}$$

and

$$t_0 = 0\,. \tag{52}$$

Differentiation of (50) yields

$$|f'(x_n)| = \exp[\int_{t_n}^{t_{n+1}} a(\tau)d\tau]\,. \tag{53}$$

So that

$$\sum_{i=0}^{N-1} \log|f'(x_n)| = \int_0^{t_N} a(\tau)d\tau\,. \tag{54}$$

Writing $t_N = N\Delta t$ yields for (54)

$$(N\Delta t)^{-1} \sum_{i=0}^{N-1} \log|f'(x_n)| = t_N^{-1} \int_0^{t_N} a(\tau)d\tau\,. \tag{55}$$

For $N \to \infty$ we see by (55) that the righthand side of (55) approaches the Lyapunov exponent χ associated with the scalar system (47) as is explained in Section 3 (Note that χ is a real constant).
As a consequence

$$\tilde{\chi}\Delta t = \lim_{N\to\infty} N^{-1} \sum_{i=0}^{N-1} \log|f'(x_n)|\,. \tag{56}$$

According to [10] the right-hand side of (56) yields the Lyapunov exponent $\tilde{\chi}$ of the time-dicscrete or iterration system (49). We see the relation

$$\chi = \tilde{\chi}\Delta t\,. \tag{57}$$

The difference in physical dimensions in χ and $\tilde{\chi}$ are thus obvious and of the usual type when converting a continuous system into a discrete system. If we set Δt = "1" then the both exponents have the same numerical value.

V. Conclusions

In the literature there are a number of definitions for the Lyapunov exponent. For three of them we have shown that they are equivalent. The proof uses dynamic eigenvalues.
These dynamic eigenvalues have real parts which can be considered to be finite time Lyanpunov exponent. They play a role in time-varying systems which is essentially the same as classical algebraic eigenvalues do in time-invariant systems.
The exponentials of the real part of the algebraic eigenvalues in invariant systems are the singular values for the fundamental matrix. This is only approximately true for the real parts of the dynamic eigenvalues in time-varying systems.
Lyapunov exponents play a certain role in systems in which chaotic behaior can occur. They provide necessery conditions. By no means these conditions are shown in literature to be sufficient. The real part of the dynamic eigenvalues gives more detailed information for these phenomena. These relation with necessery and/or sufficient conditions for chaotic processes should be studied in more detail.

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