CANONICAL REPRESENTATIONS FOR SINGLE-INPUT SINGLE-OUTPUT LINEAR TIME-VARYING SYSTEMS

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Abstract—*Scalar linear time-varying systems are addressed. In particular, canonical representations with integrators, multipliers and adders are presented. Essentially, it is shown that most of the well-known topologies for constant systems can be generalized to the time-varying context by replacing the conventional eigenvalues by the earlier introduced dynamic eigenvalues. However, one known topology can not be generalized. In stead, the adjoint system results.*

I. INTRODUCTION

As is also shown in the accompanying paper, linear time-varying (LTV) systems arise as variational models of nonlinear dynamic systems [1]. Morerover, LTV systems model a variety of physical and technical systems with time-depenent parameters [2].
On the other hand, in classical filter design the solution of linear time-invariant (LTI) systems can be obtained by using integrators, multipliers and adders, respectively. Also, for such LTI signal processors different, but equivalent representations are known ([3], [4]).
In this article, canonical representations of linear *time-varying* (LTV) systems are derived. To be more precize, possible generalizations of known LTI-topologies to the LTV-context are studied. Then, it turns out that one known LTI-topology is not suitable for such a generalization. It is also shown that two other LTI-topologies, viz. a so-called series representation and the so-called cascade representation can be generalized indeed. In the latter, the conventional algebraic eigenvalues have to be replaced by the earlier introduced *dynamic* eigenvalues ([5], [6], [7], [8]).
Related results can be found in [9] and ([10], [11]). Essentially, these contributions are based on well-known mathematical methods for the factorization of the associated polynomial differential system operator (see also [12]). In contrast, our approach uses the Riccati transformation as described in [13]. Basically, this transformation effectuates an appropriate order reduction and a subsequent decoupling of the associated LTV system equations. A successive application of the above mentioned Riccati transformation *triangularizes* the accompanying LTV system matrix step-by-step ([7], [8], [14], [15]).
In the next section, first the original scalar differential equation is rewritten in a state-space system description. Then, an associated direct signal processor representation is easily obtained.
In Section 3, first an earlier obtained alternative method for the classical Cauchy-Floquet decomposition is shortly summarized [16]. Then, the cascade representation as described in [17] follows immediately. In this representation, the *dynamic* eigenvalues play the role of time-varying multipliers. As another result, it is shown that any triangularization step produces a next representation with one extra dynamic eigenvalue as a multiplier of the LTV processor.
Finally, in Section 4 it is shown that an alternative direct realization for LTI systems does not have a LTV antipode. In stead, the adjoint system is realized.

II. A DIRECT SERIES REPRESENTATION

Consider the inhomogeneous scalar linear differential equation for the unknown $x = x(t)$ with normalized time-varying coefficients $a_i = a_i(t)$ $(i = 1, \dots, n)$

$$D^n x + a_n D^{n-1} x + \cdots + a_2 Dx + a_1 x = f\ , \quad (1)$$

in which $D = d/dt$ while $f = f(t)$ is a known function of time t. By introducing the new variables $\{x_1, x_2, \dots, x_n\}$ as

$$x_1 = x, x_2 = \dot{x}_1, x_3 = \dot{x}_2, \dots, x_n = \dot{x}_{n-1}\ , \quad (2)$$

where the dot stands for a time-derivative, we obtain from (1)

$$\dot{x}_n = -a_n x_1 - \cdots - a_1 x_n + f\ . \quad (3)$$

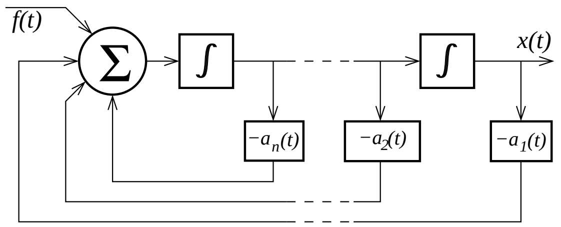

Fig. 1. A direct series representation.

On account of (2) and (3), the canonical direct series representation of figure 1 is easily recognized.
Next, introduce the n-dimensional column-vector $\boldsymbol{x}$ with $\boldsymbol{x}^T = [\, x_1, \ldots, x_n \,]$ (T stands for the transpose) and unit vectors $\boldsymbol{e}_j^{(n)}$, $(1 \le j \le n)$, with $[\boldsymbol{e}_j^{(n)}]^T = [\, \delta_{1j}, \ldots, \delta_{nj} \,]$, where δ_{ij} denotes the Kronecker symbol. Then, the partitioned $n \times n$ Frobenius companion system matrix $\mathbf{A}(t)$ is introduced as

$$\mathbf{A}(t) = \begin{bmatrix} \mathbf{I}_{n-1}^{+} & \boldsymbol{e}_{n-1}^{(n-1)} \\ -\boldsymbol{a}^T(t) & -a_1(t) \end{bmatrix} , \tag{4}$$

where $\mathbf{I}_k^{+}$ denotes the k-dimensional square shift matrix, given by

$$\mathbf{I}_k^{+} = \begin{bmatrix} 0 & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \ldots & \ldots & 0 \end{bmatrix} , \tag{5}$$

while the varying system parameters $\{a_2, a_3, \ldots, a_n\}$ are collected in the time-dependent row vector $\boldsymbol{a}^T = [\, a_n, \ldots, a_2 \,]$.
Now, the state-space description of (1) follows as

$$\dot{\boldsymbol{x}} = \mathbf{A}(t)\boldsymbol{x} + \boldsymbol{e}_n^{(n)} f(t) , \tag{6}$$

with additional output equation

$$x = x_1 . \tag{7}$$

As a result, equations (6) and (7) are represented by the LTV signal processor in figure 1.

III. The Cascade and Hybrid Representation

As we showed earlier [8], there exists a lower triangular Riccati transformation matrix $\mathbf{R}$

$$\boldsymbol{x}(t) = \mathbf{R}(t)\boldsymbol{y}(t) , \tag{8}$$

by which system (6) is transformed into the new system

$$\dot{\boldsymbol{y}} = \{\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) + \mathbf{I}_n^{+}\}\boldsymbol{y} + \boldsymbol{e}_n^{(n)} f \tag{9}$$

with $\boldsymbol{y}^T = [y_1, \ldots, y_n]$, while the output equation (7) goes into

$$x = y_1 . \tag{10}$$

Now, it is easily observed that the original differential equation (1) is given by the Cauchy-Floquet decomposition [16]

$$(D - \lambda_n(t))(D - \lambda_{n-1}(t)) \ldots (D - \lambda_1(t))x = f(t) . \tag{11}$$

Secondly, this decomposition constitutes the canonical cascade signal processor as depicted in figure 2. Note, that its mulptipliers $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ may be complex valued functions of the time t.

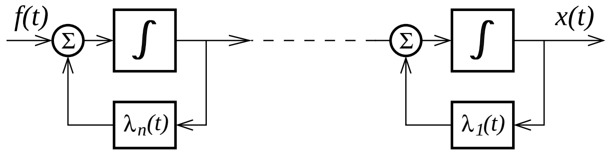

Fig. 2. The cascade toplogy.

As we explained earlier ([7], [8], [14]), each triangularization step needs a *particular* solution of a vector Riccati differential equation. If $p_1, p_2, \ldots,$ and, p_{n-1} denotes the components of the solution vector of the first Riccati-equation, it is easily shown that the topology depicted in figure 3 is equivalent to the LTV signal processor of figure 1.

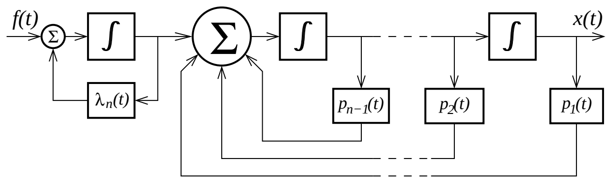

Fig. 3. A hybrid topology after one Riccati transformation.

The system matrix of system (9) indeed confirms that the functions $\lambda_i(t)$ are a kind of eigenvalues. To show this regorously, consider the homogeneous equation of system (9), thus $f \equiv 0$. This equation is investigated for modal solutions of the form

$$\boldsymbol{y}_j(t) = \boldsymbol{u}_j(t) \exp[\gamma_j(t)] \tag{12}$$

in which

$$\gamma_j(t) = \int_0^t \lambda_j(\tau) d\tau \tag{13}$$

and

$$\boldsymbol{u}_j(t) = [\, u_{1,j}(t), \ldots, u_{j-1,j}(t), 1, 0, \ldots, 0 \,]^T . \tag{14}$$

Substitution of (12) and (14) in (9) with $f \equiv 0$, yields that (12) is indeed a solution, only if $\boldsymbol{u}_j$ satisfies

$$\dot{\boldsymbol{u}}_j = [\{\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) + \mathbf{I}_n^{+}\} - \lambda_j \mathbf{I}_n]\boldsymbol{u}_j , \tag{15}$$

where $\mathbf{I}_n$ is the n-dimensional unity matrix. For a linear time-invariant system, $\boldsymbol{u}_i(t)$ and $\lambda_i(t)$ are constants and, as a consequence, the left hand side of (15) becomes zero. Hence, the classical eigenvalue problem results. As we argued earlier, this justifies to call $\lambda_i(t)$ a *dynamic* eigenvalue and $\boldsymbol{u}_i(t)$ a *dynamic* eigenvector ([5], [6], [7], [8]).
Now, it is clear that

$$\mathbf{U}(t) = [\ \boldsymbol{u}_1(t), \ldots, \boldsymbol{u}_n(t)\ ] \tag{16}$$

is a transformation matrix that transforms with

$$\boldsymbol{y}(t) = \mathbf{U}(t)\boldsymbol{z}(t) \tag{17}$$

the homogeneous equation (10) into

$$\dot{\boldsymbol{z}} = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)\boldsymbol{z}\ . \tag{18}$$

Hence, the fundamental matrix $\boldsymbol{\Phi} = \boldsymbol{\Phi}(t)$ of (1) is obtained as

$$\boldsymbol{\Phi} = \mathbf{R}\mathbf{U}\ \text{diag}\{\exp(\gamma_1), \exp(\gamma_2), \ldots, \exp(\gamma_n)\}\ , \tag{19}$$

in which $\gamma_j = \gamma_j(t)$ is given by (13).

IV. An alternative configuration

Finally, consider the signal processor topology of figure 4.

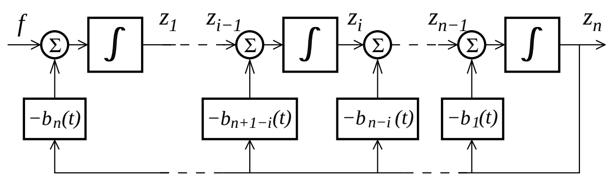

Fig. 4. An second series realization.

The set of associated equations are obtained as

$$\dot{z}_1 = -b_n z_n + f\ , \tag{20}$$

$$\dot{z}_i = z_{i-1} - b_{n+1-i} z_n \quad (i = 2, \ldots, n)\ . \tag{21}$$

Equation (21) yields after $(i-1)$ differentations

$$D^i z_i = D^{i-1} z_{i-1} - D^{i-1}(b_{n+1-i} z_n) \quad (i = 2, \ldots, n). \tag{22}$$

Adding all equations in (22), we obtain

$$D^n z_n + \sum_{l=1}^{n} D^{n-l}(b_l z_n) = f\ . \tag{23}$$

Since the Leibniz-rule of differentiation gives

$$D^{n-l}(b_l z_n) = \sum_{k=0}^{n-l} \binom{n-l}{k} D^{n-l-k} b_l D^k z_n\ , \tag{24}$$

equation (23) can be rewritten as

$$D^n z_n + \sum_{k=0}^{n-1} \left[\sum_{l=1}^{n-k} \binom{n-l}{k} D^{n-l-k} b_l \right] D^k z_n = f\ . \tag{25}$$

Next, we conclude that if

$$a_{n-l} = \sum_{l=1}^{n-k} \binom{n-l}{k} D^{n-l-k} b_l\ , \tag{26}$$

while

$$z_n = x\ , \tag{27}$$

then (25) is equivalent to (1).
Now, it is observed that for LTI systems equation (26) reduces to

$$a_{n-k} = b_{n-k}\ . \tag{28}$$

Then, and only then, the realizations in figure 1 and figure 4 are equivalent.
Finally, a small change in the realization scheme of figure 4 is applied, see figure 5 as a result. There we have

$$\left.\begin{aligned} \dot{z}_1 &= b_n z_n - f \\ &\vdots \\ \dot{z}_i &= -z_{i-1} + b_{n+1-i} z_n \\ &\vdots \\ \dot{z}_n &= -z_{n-1} + b_1 z_n \end{aligned}\right\}\ . \tag{29}$$

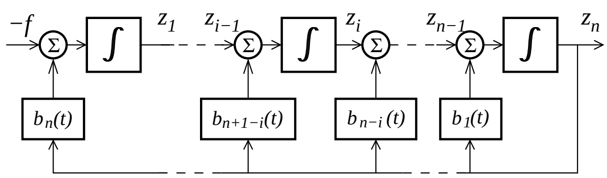

Fig. 5. The series realization represents the adjoint system.

This yields the following collection of equations

$$\left.\begin{aligned} &(-1)^n D^n(z_n) + (-1)^{n-1} D^{n-1}(b_1 z_n) = \\ &\qquad\qquad (-1)^{n-1} D^{n-1} z_{n-1} \\ &\vdots \\ &(-1)^2 D^2(z_2) + (-1)^1 D^1(b_{n-1} z_n) = (-1)^1 D^1 z_1 \\ &(-1)^1 D^1(z_1) + (-1)^0 D^0(b_n z_n) = f\ , \end{aligned}\right\} \tag{30}$$

As a consequence, it follows that

$$\begin{aligned} &(-1)^n D^n(z_n) + (-1)^{n-1} D^{n-1}(b_1 z_n) + \cdots + \\ &+ (-1)^1 D^1(b_{n-1} z_n) + (-1)^0 D^0(b_n z_n) = f \end{aligned} \tag{31}$$

Thus, the second series scheme of figure 5 represents the adjoint equation of (1) iff

$$b_i = a_i \; . \tag{32}$$

V. Conclusions

A variety of canonical representations for LTV signal processors is derived. They are realized with integrators, time-dependent multipliers and adders, respectively. Most of the proposed processors are generalizations of known topologies for constant systems. It turns out that the classical eigenvalues in the time-invariant context have to be replaced by the earlier introduced dynamic eigenvaleus.

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