DYNAMIC EIGENVALUES AND LYAPUNOV EXPONENTS FOR NONLINEAR CIRCUITS

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Abstract— *The Lyapunov exponents associated with nonlinear electronic circuits are related to the earlier introduced dynamic eigenvalues.*

I. Introduction

Electronic circuits are nonlinear by nature. Roughly speaking, two different kinds of operation can be distinguished. Those which are described by the behavior of small signals around a fixed operating point, like class A amplifiers, and those that behave in a time-varying mode of operation, like oscillators.

As is well-known, Lyapunov exponents characterize the time-behavior of small variations in these circuits. Therefore, the concept of a variational circuit is introduced. It has the same topology as the original nonlinear circuit, while each nonlinear element is replaced by a differential one, evaluated along the mode of operation [1]. Then, fixed operation point circuits result into the well-known linear and constant small-signal circuits, while the time-varying mode of operation circuits give rise to linear time-varying (LTV) variational circuits, respectively.

In an ealier contribution, we introduced the tableau method to prove this statement for a broad class of branch relations [2]. In this paper, the tableau method is used again, but now for completely general constitutive branch relations. Moreover, it is shown that the LTV system equation for variations in the state variables is arbitrarily obtained directly from the original nonlinear circuit or from the LTV circuit, respectively.

Next, it is noticed that LTV systems produce modal solutions ([3], [4], [5]). These solutions generalize the well-known modes pertaining to linear time-invariant (LTI) systems [6]. In particular, the arguments of the exponential functions involved, generalize the arguments of the exponentials in the traditional LTI case [7]. This leads to the concept of dynamic eigenvalues as generalizations of the classical algebraic eigenvalues.

Since the traditional eigenvalues also define Lyapunov exponents as their generalization, it is worthwhile to investigate whether there is a relation between the dynamic eigenvalues and the Lyapunov exponents.

To that aim, we first shortly describe how the modal solutions for any arbitrarily LTV system are obtained. Next, the system matrix is transformed into a triangular form which is subsequently solved analytically. Then, it becomes clear that the Lyapunov exponents are the mean values of the dynamic eigenvalues over an infinite interval of time. This can be seen in several ways. Each way reflects a way of defining the set of Lyapunov exponents. For example, in [8] the Lyapunov exponents are defined with the principal axes of the ellipsoid associated with the product of the fundamental matrix of the LTV system and its conjugate transpose. On its turn, the fundamental matrix is based on the Jacobian of the state equation of the nonlinear system associated with the nonlinear circuit.

However, in this paper the definition in [9] is followed. There, the Lyapunov exponents are directly related to the associated variational solutions of the original nonlinear system. Finally, in [10] it is shown that both definitions are equivalent.

II. The tableau equations

We consider nonlinear electrical circuits with n nodes and b branches. The time behavior of the circuit is described completely by the branch currents and voltages, respectively. Their solution is contained in the circuit tableau as it collects all the relevant information of the circuit, viz. the Kirchhoff laws (KCL's and KVL's), the constitutive relations of the constituent elements, together with the conservation laws of charge and flux, respectively.

For $(n-1)$ nodes there is an independent current law which can be combined into

$$\mathbf{A}\boldsymbol{i} = \boldsymbol{0} \; , \qquad (1)$$

where $\mathbf{A}$ is the $(n-1) \times b$ reduced node-branch incidence matrix with rank $(n-1)$. In (1), $\boldsymbol{i} = \boldsymbol{i}(t) \in \mathbb{R}^b$

denotes the set of time dependent branch currents.
Next, the KVL equations are written as

$$\boldsymbol{u} - \mathbf{A}^T \boldsymbol{v} = \boldsymbol{0} \ , \tag{2}$$

in which the vectors $\boldsymbol{u} = \boldsymbol{u}(t) \in \mathbb{R}^b$ and $\boldsymbol{v} = \boldsymbol{v}(t) \in \mathbb{R}^{n-1}$ denote the branch voltages and the node voltages with respect to a chosen datum,respectively (T stands for the transpose).

Furthermore, it is assumed that the circuit has ρ (two-terminal) resistors and independent sources, γ capacitors and $\lambda = b - \rho - \gamma$ inductors. For the nonlinear resistors, including the sources, the branch relations are taken as

$$f_R(u_k, i_k) = 0 \quad (k = 1, 2, \ldots, \rho) \tag{3}$$

for each combination (u_k, i_k) of a branch current and voltage. Here, the symbol f denotes a nonlinear function description. Then, the relation between u_k and i_k (both functions of the time t) in (3) is completely general [11].

For the nonlinear capacitors we have the equally general constitutive relations

$$f_C(u_k, q_k) = 0 \quad (k = \rho+1, \rho+2, \ldots, \rho+\gamma) \tag{4}$$

for each combination (u_k, q_k) of a time-dependent capacitor voltage and charge. Moreover, the nonlinear inductors are defined by

$$f_L(i_k, \phi_k) = 0 \quad (k = \rho+\gamma+1, \rho+\gamma+2, \ldots, b) \tag{5}$$

for each combination (i_k, ϕ_k) of inductor current and flux, again both time-dependent.

Finally, associated with the γ capacitors and λ inductors there are γ, respectively λ laws of conservation, namely

$$i_k = \frac{dq_k}{dt} \quad (k = \rho+1, \rho+2, \ldots, \rho+\gamma) \tag{6}$$

and

$$u_k = \frac{d\phi_k}{dt} \quad (k = \rho+\gamma+1, \rho+\gamma+2, \ldots, b) \ . \tag{7}$$

Totally, there are $2b + \gamma + \lambda + (n-1)$ independent equations in (1)-(7) for an equal number of unknowns. In order to arrive at a differential equation for the charge and the flux vectors $\boldsymbol{q} = \boldsymbol{q}(t)$ and $\boldsymbol{\phi} = \boldsymbol{\phi}(t)$, respectively, $\boldsymbol{u}$, $\boldsymbol{i}$ and $\boldsymbol{v}$ are eliminated from the circuit tableau (1)-(7). The result is written as the nonlinear state-equation

$$\left.\begin{aligned} \dot{\boldsymbol{q}} &= \boldsymbol{f}_Q(\boldsymbol{q}, \boldsymbol{\phi}, \boldsymbol{e}) \\ \dot{\boldsymbol{\phi}} &= \boldsymbol{f}_\Phi(\boldsymbol{q}, \boldsymbol{\phi}, \boldsymbol{e}) \end{aligned}\right\} \ , \tag{8}$$

where the source contribution vector $\boldsymbol{e}$ is taken as a seperate variable.

Now, if $\boldsymbol{q}_s$ and $\boldsymbol{\phi}_s$ are small deviations from a solution of (8), originated by an equally small source contribution $\boldsymbol{e}_s$ of $\boldsymbol{e}$, then the following variational equation is easily obtained

$$\frac{d}{dt}\begin{bmatrix} \boldsymbol{q}_s \\ \boldsymbol{\phi}_s \end{bmatrix} = \mathbf{A}_s \begin{bmatrix} \boldsymbol{q}_s \\ \boldsymbol{\phi}_s \end{bmatrix} + \mathbf{B}_s \boldsymbol{e}_s \ , \tag{9}$$

where $\mathbf{A}_s = \mathbf{A}_s(t)$ collects the Jacobians of $\boldsymbol{f}_Q$ and $\boldsymbol{f}_\Phi$ with respect to $\boldsymbol{q}$ and $\boldsymbol{\phi}$, respectively, while $\mathbf{B}_s = \mathbf{B}_s(t)$ collects the Jacobians of $\boldsymbol{f}_Q$ and $\boldsymbol{f}_\Phi$ with respect to $\boldsymbol{e}_s$ [12].

An alternative way to arrive at (9) is presented next by taking two possible solutions $\{\boldsymbol{I},\ \boldsymbol{U},\ \boldsymbol{V},\ \boldsymbol{Q},\ \boldsymbol{\Phi}\}$ and $\{\boldsymbol{I} + \boldsymbol{i}_s,\ \boldsymbol{U} + \boldsymbol{u}_s,\ \boldsymbol{V} + \boldsymbol{v}_s,\ \boldsymbol{Q} + \boldsymbol{q}_s,\ \boldsymbol{\Phi} + \boldsymbol{\phi}_s\}$ of (1)-(7). Here, the subscript s again refers to small variations with respect to the first solution. Note that these solutions generally are time-dependent.

In view of (3) (4) and (5), it follows respectively

$$f_R(U_k + u_{s,k}, I_k + i_{s,k}) - f_R(U_k, I_k) = 0 \qquad (k = 1, \ldots, \rho) \ , \tag{10}$$

$$f_C(U_k + u_{s,k}, Q_k + q_{s,k}) - f_C(U_k, Q_k) = 0 \qquad (k = \rho+1, \ldots, \rho+\gamma) \tag{11}$$

and

$$f_L(I_k + i_{s,k}, \Phi_k + \phi_{s,k}) - f_L(I_k, \Phi_k) = 0 \qquad (k = \rho+\gamma+1, \ldots, b) \ . \tag{12}$$

Since $\boldsymbol{u}_s$, $\boldsymbol{i}_s$, $\boldsymbol{v}_s$, $\boldsymbol{q}_s$ and $\boldsymbol{\phi}_s$ are small compared with $\boldsymbol{U}$, $\boldsymbol{I}$, $\boldsymbol{V}$, $\boldsymbol{Q}$ and $\boldsymbol{\Phi}$, these equations are approximated by a first order Taylor series expansion. This yields

$$\frac{\partial f_R(U_k, I_k)}{\partial u_k} u_{s,k} + \frac{\partial f_R(U_k, I_k)}{\partial i_k} i_{s,k} = 0 \qquad (k = 1, 2, \ldots, \rho) \ , \tag{13}$$

$$\frac{\partial f_C(U_k, Q_k)}{\partial u_k} u_{s,k} + \frac{\partial f_C(U_k, Q_k)}{\partial q_k} q_{s,k} = 0 \qquad (k = \rho+1, \ldots, \rho+\gamma) \tag{14}$$

and

$$\frac{\partial f_L(I_k, \Phi_k)}{\partial i_k} i_{s,k} + \frac{\partial f_L(I_k, \Phi_k)}{\partial \phi_k} \phi_{s,k} = 0 \qquad (k = \rho+\gamma+1, \ldots, b) \ . \tag{15}$$

Now, it is noted that (13), next to voltage and current sources, also define LTV resistors with resistance $R_k = R_k(t)$, given by

$$R_k = -\left(\frac{\partial f_R}{\partial i_k}\right)\left(\frac{\partial f_R}{\partial u_k}\right)^{-1} . \quad (16)$$

Also, (14) and (15) define LTV capacitors inductors, respectively, with capacitance $C_k = C_k(t)$, given by

$$C_k = -\left(\frac{\partial f_C}{\partial u_k}\right)\left(\frac{\partial f_C}{\partial q_k}\right)^{-1} \quad (17)$$

and inductance $L_k = L_k(t)$, given by

$$L_k = -\left(\frac{\partial f_L}{\partial i_k}\right)\left(\frac{\partial f_L}{\partial \phi_k}\right)^{-1} . \quad (18)$$

Finally, we have with (1)

$$\mathbf{A}\boldsymbol{i}_s = \boldsymbol{0} \quad (19)$$

and with (2)

$$\boldsymbol{u}_s - \mathbf{A}^T\boldsymbol{v}_s = \boldsymbol{0} , \quad (20)$$

while it follows from (6) and (7)

$$i_{s,k} = \frac{dq_{s,k}}{dt} \quad (k = \rho+1, \ldots, \rho+\gamma) \quad (21)$$

and

$$u_{s,k} = \frac{d\phi_{s,k}}{dt} \quad (k = \rho+\gamma+1, \ldots, b) , \quad (22)$$

respectively.

The equations (16) - (22) define the LTV variational circuit associated with the original nonlinear circuit. And, since the circuit matrix $\mathbf{A}$ in (1) and (2) equals that in (19) and (20) respectively, it is concluded that its topology equals the topology of the original circuit. As before, again the variables $\boldsymbol{u}_s$, $\boldsymbol{i}_s$ and $\boldsymbol{v}_s$ are eliminated from the circuit tableau. And, because of the identical topology of the original and variational circuit, it follows that the elimination steps to arrive at the state-equation (8) are also identical. Moreover, since the nonlinear elements are replaced by (linearized) incremental ones, we directly obtain the linearized form (9) of (8) (see also [13]).

III. The Lyapunov exponents of the nonlinear circuit

In this section our starting point is the homogeneous state equation (9), here rewritten as

$$\dot{\boldsymbol{x}}(t) = \mathbf{A}_s(t)\boldsymbol{x}(t) , \quad (23)$$

in which the dimension n of the system matrix $\mathbf{A}_s$ now refers to the number of independent states.

It is possible to derive a state space transformation ([3], [4])

$$\boldsymbol{x}(t) = \mathbf{R}(t)\boldsymbol{y}(t) , \quad (24)$$

such that

$$\dot{\boldsymbol{y}}(t) = \mathbf{B}(t)\boldsymbol{y}(t) , \quad (25)$$

with $\mathbf{B}(t)$ an upper triangular matrix

$$\mathbf{B}(t) = \begin{bmatrix} \lambda_1(t) & & b_{ij}(t) \\ & \ddots & \\ 0 & & \lambda_n(t) \end{bmatrix} . \quad (26)$$

Here, $\{\lambda_1(t), \ldots, \lambda_n(t)\}$ is the set of dynamic eigenvalues of (23). Using the transformation

$$\boldsymbol{y}(t) = \mathbf{F}(t)\mathrm{Diag}\left\{\int_{t_0}^{t} \lambda_i(\tau)d\tau\right\}\boldsymbol{y}(t_0) \quad (27)$$

with t_0 the initial time and

$$\mathbf{F}(t) = \begin{bmatrix} 1 & & f_{ij}(t) \\ & \ddots & \\ 0 & & 1 \end{bmatrix} , \quad (28)$$

we obtain an analytic expresson for the solution of (25). Substitution of (27) into (25) shows that $\mathbf{F}(t)$ has to satisfy

$$\left.\begin{array}{c} \dot{\mathbf{F}} + \mathbf{F}\boldsymbol{\Lambda} = \mathbf{B}\mathbf{F} \\ \mathbf{F}(t_0) = \mathbf{I} \end{array}\right\} , \quad (29)$$

where

$$\boldsymbol{\Lambda} = \mathrm{Diag}\{\lambda_1(t), \ldots, \lambda_n(t)\} . \quad (30)$$

For the elements $f_{ij}(t)$ $(i < j)$ of $\mathbf{F}$ in (29) the set of differential equations is given by

$$\dot{f}_{ij} + f_{ij}\lambda_j = \lambda_i f_{ij} + \sum_{k=1}^{j-i-1} b_{i,i+k}f_{i+k,j} + b_{ij} . \quad (31)$$

Thus f_{12}, f_{23}, $\ldots$ $f_{n-1,n}$ are solved first. Then f_{13}, f_{24}, $\ldots$ $f_{n-2,n}$ follows and finally $f_{1,n}$ is found.

As a consequence, the solution of (23) can be written as

$$\boldsymbol{x}(t) = \boldsymbol{\Phi}(t,t_0)\boldsymbol{x}(t_0) , \quad (32)$$

where the normalized fundamental matrix $\boldsymbol{\Phi}(t,t_0)$ is given by

$$\boldsymbol{\Phi}(t,t_0) = \boldsymbol{\Phi}(t)\boldsymbol{\Phi}^{-1}(t_0) , \quad (33)$$

while $\boldsymbol{\Phi}(t)$ equals

$$\boldsymbol{\Phi}(t) = \mathbf{R}(t)\mathbf{F}(t)\mathrm{Diag}[\exp\{\gamma_k(t)\}] , \quad (34)$$

in which

$$\gamma_k(t) = \int_{t_0}^{t} \lambda_k(\tau) d\tau \ . \qquad (35)$$

The solution (32) can also be written as the modal expansion ([3], [4], [6])

$$\boldsymbol{x}(t) = \sum_{i=1}^{n} \boldsymbol{u}_i(t) \exp\{\gamma_i(t)\} x_i(t_0) \ , \qquad (36)$$

where $\boldsymbol{u}_i$ $(i = 1, 2, \dots, n)$ is the i-th column of the matrix product $\mathbf{RF}$ (34).

We now introduce the mean value $\langle\lambda_k\rangle$ of $\lambda_k(t)$ as

$$\langle\lambda_k\rangle = \lim_{t\to\infty} t^{-1}\gamma_k(t) = \lim_{t\to\infty} t^{-1} \int_{t_0}^{t} \lambda_k(\tau) d\tau \qquad (37)$$

and an ordening such that

$$\mathrm{Re}\langle\lambda_1\rangle > \mathrm{Re}\langle\lambda_k\rangle \quad (k = 2, \dots, n) \ . \qquad (38)$$

Then we have [10]

$$\lim_{t\to\infty} \exp[\{\lambda_k(t) - \lambda_1(t)\}t] = 0 \quad (k \neq 1) \qquad (39)$$

and

$$\lim_{t\to\infty} \exp[\gamma_k(t) - \langle\lambda_k\rangle t] = 1 \quad \forall k \ . \qquad (40)$$

Furthermore, we observe from (36) that for every component $x_i(t)$ we have an amplitude $A_i(t)$ such that

$$x_i(t) \to A_i(t) \exp\{\langle\lambda_1\rangle t\} \quad \text{if} \quad t \to \infty \ . \qquad (41)$$

If this function has a regular behavior, then we find

$$\mathrm{Re}\langle\lambda_1\rangle = \lim_{t\to\infty} t^{-1} |x_i(t)| \ . \qquad (42)$$

This proofs along the lines of [9] that $\mathrm{Re}\langle\lambda_1\rangle$ equals a Lyapunov exponent.

IV. Conclusions

By using the tableau method, it is proved that associated with each nonlinear circuit there is a LTV circuit with the same topology as the original one. It describes the time behavior of small variations in voltages and currents of the original circuit. Next, the solution of the variational circuit is described in terms of the earlier introduced dynamic eigenvalues. Finally, the Lyapunov exponents follow as the mean values of these dynamic eigenvalues.

References

[1] T.E. Stern, *Theory of Nonlinear Networks and Systems, an Introduction*, Addison-Wesley, 1965, pp. 314 and 234.

[2] F.L. Neerhoff, P. van der Kloet, A. van Staveren and C.J.M. Verhoeven, *Time-Varying Small Signal Circuits for Nonlinear Electronics*, Proc. Nonlinear Dynamics of Electronic Systems (NDES) 2000, Catania, Italy, May 18-20, 2000, pp. 81-84.

[3] P. van der Kloet and F.L. Neerhoff, *Modal Factorization of Time-Varying Models for Nonlinear Circuits by the Riccati Transform*, Proc. IEEE International Symposium on Circuits and Systems (ISCAS) 2001, Sydney, Australia, May 2001, pp. III-553-556.

[4] F.L. Neerhoff and P. van der Kloet, *A Complementary View on Time-Varying Systems*, Proc. ISCAS 2001, Sydney, Australia, May 2001, pp. III-779-782.

[5] P. van der Kloet and F.L. Neerhoff, *The Riccati Equation as Characteristic Equation for General Linear Dynamic Systems*, Nonlinear Theory and its Applications (NOLTA) 2001, Miyagi, Zao, Japan, 28 Oct - 1 Nov, 2001, Vol. 2, pp. 425-428.

[6] P. van der Kloet, *Modal Solutions for Linear Time-Varying Systems*, Doctor Thesis, Delft Techn. Univers., Delft, The Netherlands, Sept. 2002, ISBN 90-901586-X.

[7] B. van der Pol, *The Fundamental Principles of Frequency Modulation*, Journ. IEEE, Vo. 93, Part III, No. 23, May, 1946, pp. 153-158.

[8] M.P. Kennedy, *Basic Concepts of Nonlinear Dynamics and Chaos*, In: Circuit and Systems Tutorials, IEEE ISCAS '94 (Ed. C. Toumazou), 1994, pp. 289-313.

[9] J.M.T. Thompson and H.B. Stewart, *Nonlinear Dynamics and Chaos*, John Wiley and Sons, Chicester, 1998.

[10] P. van der Kloet and F.L. Neerhoff, *Lyapunov Exponents and Dynamical Eigenvalues*, XII Intern. Symp. on Theor. Electr. Eng. (ISTET'03), Warsaw, Poland, July 6-9, 2003.

[11] L.O. Chua, C.A. Desoer and E.S. Kuh, *Linear and Nonlinear Circuits*, McGraw-Hill, 1987.

[12] W.J. Rugh, *Linear System Theoty*, Prentice Hall, 1991.

[13] L.O. Chua and P. Lin, *Computer-Aided Analysis of Electronic Circuits*, Prentice Hall, 1975.