Dynamic Eigenvalues for Scalar Linear Time-Varying Systems

P. van der Kloet and F.L. Neerhoff
Department of Electrical Engineering
Delft University of Technology
Mekelweg 4
2628 CD Delft
The Netherlands

Abstract

In this paper, an algorithm is derived for computing the earlier introduced eigenvalues of scalar varying systems. These new types of eigenvalues are key quantities for describing the dynamic behavior of such systems. They generalize the conventional antipodes pertaining to constant systems. Essentially, the algorithm performs successive Riccati transformations that gradually triangularize the accompanying time-varying system matrix.

1 Introduction

Any solution of a set of first order linear homogeneous differential equations with *time-varying* coefficients can be written as the sum of elementary modes. Each individual mode is the product of a (time-varying) amplitude and an exponential phase function. The argument of the exponent is a time-integral over a so-called dynamic eigenvalue [1, 2].
In this paper, an algorithm for these dynamic eigenvalues pertaining to scalar linear differential equations is developed. As such, the work of Kamen [3] on second order equations is generalized. Also, the work of Zhu [4] is reformulated (*cf.* [5]). This work is fundamentally based on the classical Cauchy-Floquet decomposition [6] of the associated scalar differential operator. Then, differential equations for the dynamic eigenvalues are obtained. At this place, it is also noted that the dynamic eigenvalues just equal the roots of Amitsur [7].
In contrast, we start with a state-space description. Then, with the aid of successive Riccati transformations [8] the afore mentioned Cauchy-Floquet decomposition follows alternatively. After each individual Riccati transformation, a next dynamic eigenvalue is obtained. At the same time, a state-space description with one dimension lower follows. Noteworthly, the structure of the successive companion matrices is maintained during the whole iteration procedure.
Also, each step asks for a *particular solution* of a Riccati differential equation. These Riccati equations appear to be the generalization of the classical characteristic equation for linear time-invariant systems [9]. As linear *time-varying* (LTV) systems are concerned, characteristic equations can be obtained using the Riccati equations together with the expressions for

the dynamical eigenvalues. This is worked out in [4] and [10] for third order systems. Of course, the results presented here must be in agreement with the results obtained in [4] and [10]. This is verified for the classical Euler differential equation [11].

2 Scalar LTV-systems

Consider a homogeneous scalar dynamic LTV-system, described by a n-th order linear differential equation with time-varying coefficients

$$D^n x + a_1(t)D^{n-1}x + \dots a_{n-1}(t)Dx + a_n(t)x = 0 \quad , \tag{2.1}$$

where $D = d/dt$ is the differential operator and $x = x(t)$ the output of the system. In state-space description, this is equivalent with

$$\dot{\boldsymbol{x}}(t) = \mathbf{A}(t)\boldsymbol{x}(t) = \begin{bmatrix} \mathbf{I}^+_{n-1} & \boldsymbol{e}_{n-1} \\ \boldsymbol{q}_n^T(t) & \tilde{q}_n(t) \end{bmatrix} \boldsymbol{x}(t) \quad . \tag{2.2}$$

Here, $\boldsymbol{q}_n^T(t)$ and $\tilde{q}_n(t)$ represent the coefficients of the differential equation, namely

$$\boldsymbol{q}_n^T(t) = -[\, a_n(t) \ \dots \ a_2(t) \,] \quad , \tag{2.3}$$

$$\tilde{q}_n(t) = -a_1(t) \quad . \tag{2.4}$$

Moreover, $\mathbf{I}^+_{n-1}$ represents an upward shift of the diagonal in an $(n-1)\times(n-1)$ unity matrix $\mathbf{I}_{n-1}$, thus

$$\mathbf{I}^+_{n-1} = \begin{bmatrix} 0 & 1 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \dots & \dots & 0 \end{bmatrix} \quad . \tag{2.5}$$

Further , $\boldsymbol{e}_{n-1}$ denotes the $(n-1)$ dimensional unitvector

$$\boldsymbol{e}_{n-1} = [\, 0, \dots, 0, 1 \,]^T \quad , \tag{2.6}$$

where T stands for the transpose. Since (2.1) and (2.2) are equivalent, we have

$$x = x_1 \quad , \tag{2.7}$$

where x_1 is the first component of $\boldsymbol{x}$.
Next, we will establish the existence and the construction of a compound state transformation

$$\boldsymbol{x} = \mathbf{R}(t)\boldsymbol{y} \quad , \tag{2.8}$$

with

$$\mathbf{R} = \mathbf{R}^{(n)}\mathbf{R}^{(n-1)} \dots \mathbf{R}^{(2)} \quad \text{and} \quad \forall \mathbf{R}^{(k)} \in \mathbb{R}^{n\times n} \quad , \tag{2.9}$$

such that we obtain

$$\dot{\boldsymbol{y}} = \mathbf{B}(t)\boldsymbol{y} \quad , \tag{2.10}$$

with $\mathbf{B} \in \mathbb{R}^{n\times n}$ an *upper triangle* matrix.
Note that $\mathbf{R}$ is lower triangular if all $\mathbf{R}^{(k)}$ are lower triangular. Moreover, note that all elements on the main diagonal of $\mathbf{R}$ equal unity if all elements of all $\mathbf{R}^{(k)}$ on the main diagonal are unity. In that case (2.7) and (2.8) will yield

$$x = y_1 \quad . \tag{2.11}$$

We will show that the transformation process is described by

$$\mathbf{A}^{(n)} = \mathbf{A} \quad , \tag{2.12}$$

$$\mathbf{A}^{(k-1)} = \left[\mathbf{R}^{-1}\mathbf{A}\mathbf{R} - \mathbf{R}^{-1}\dot{\mathbf{R}}\right]^{(k)} \quad (k = n, n-1, \ldots, 2) \quad , \tag{2.13}$$

where $[\quad]^{(k)}$ denotes that all matrices within the brackets have the superscript (k). This process finally yields

$$\mathbf{B} = \mathbf{A}^{(1)} \quad , \tag{2.14}$$

where $\mathbf{B}$ is the triangular matrix in (2.10).
To show that $\mathbf{B}$ is triangular, assume that $\mathbf{A}^{(k)}$ has the following form

$$\mathbf{A}^{(k)} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{0} & \mathbf{A}_{22} \end{bmatrix}^{(k)} \quad , \tag{2.15}$$

with $\mathbf{A}_{11}^{(k)} \in \mathbb{R}^{k\times k}$, $\mathbf{A}_{12}^{(k)} \in \mathbb{R}^{k\times(n-k)}$ and $\mathbf{A}_{22}^{(k)} \in \mathbb{R}^{(n-k)\times(n-k)}$ being a triangular matrix.
Assume moreover that

$$\mathbf{R}^{(k)} = \begin{bmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}^{(k)} \tag{2.16}$$

has the same partitioning as $\mathbf{A}^{(k)}$. Here $\mathbf{I}$ denotes the identity matrix.
The substitution of (2.15) and (2.16) into (2.13) yield

$$\mathbf{A}^{(k-1)} = \begin{bmatrix} \mathbf{P}^{-1}\mathbf{A}_{11}\mathbf{P} - \mathbf{P}^{-1}\dot{\mathbf{P}} & \mathbf{P}^{-1}\mathbf{A}_{12} \\ \mathbf{0} & \mathbf{A}_{22} \end{bmatrix}^{(k)} \quad . \tag{2.17}$$

It is clear from (2.17) that if the last row of $[\mathbf{P}^{-1}\mathbf{A}_{11}\mathbf{P} - \mathbf{P}^{-1}\dot{\mathbf{P}}]^{(k)}$ equals zero with an exception for the last element in this row, then $\mathbf{A}^{(k-1)}$ can be partitioned as (2.15) with $\mathbf{A}_{22}^{(k-1)}$ triangular.
If we assume for $\mathbf{P}$ in (2.16) the form

$$\mathbf{P} = \begin{bmatrix} \mathbf{I} & \boldsymbol{0} \\ \boldsymbol{p}^T & 1 \end{bmatrix}^{(k)} \tag{2.18}$$

with $\boldsymbol{p}^T \in \mathbb{R}^{1\times(k-1)}$ and for $\mathbf{A}_{11}$ in (2.15) the same partitioning as $\mathbf{P}$

$$\mathbf{A}_{11} = \begin{bmatrix} \mathbf{D} & \boldsymbol{b} \\ \boldsymbol{c}^T & d \end{bmatrix}^{(k)} \quad , \tag{2.19}$$

with $\mathbf{D} \in \mathbb{R}^{(k-1)\times(k-1)}$, $\boldsymbol{b} \in \mathbb{R}^{(k-1)\times 1}$, $\boldsymbol{c}^T \in \mathbb{R}^{1\times(k-1)}$ then we will find

$$\left[\mathbf{P}^{-1}\mathbf{A}_{11}\mathbf{P} - \mathbf{P}^{-1}\dot{\mathbf{P}}\right]^{(k)} = \begin{bmatrix} \mathbf{D} + \boldsymbol{b}\boldsymbol{p}^T & \boldsymbol{b} \\ \boldsymbol{0}^T & \lambda \end{bmatrix}^{(k)} \tag{2.20}$$

if and only if $\boldsymbol{p}^T$ satisfies

$$\dot{\boldsymbol{p}}^T = -\boldsymbol{p}^T\boldsymbol{b}\boldsymbol{p}^T - \boldsymbol{p}^T\mathbf{D} + d\boldsymbol{p}^T + \boldsymbol{c}^T \tag{2.21}$$

and

$$\lambda = d - \boldsymbol{p}^T\boldsymbol{b} \quad . \tag{2.22}$$

The equation (2.20) shows that $[\mathbf{P}^{-1}\mathbf{A}_{11}\mathbf{P} - \mathbf{P}^{-1}\dot{\mathbf{P}}]^{(k)}$ has the desired entries in its last row. Repetitions of this procedure will finally yield the triangular matrix

$$\mathbf{B} = \begin{bmatrix} \lambda_1 & \cdots & \cdots & \vdots \\ \vdots & \ddots & b_{ij} & \vdots \\ \vdots & \mathbf{0} & \ddots & \vdots \\ \vdots & \cdots & \cdots & \lambda_n \end{bmatrix} \quad . \tag{2.23}$$

For the scalar LTV system (2.1) or its state space equivalent (2.2) the procedure described here will be given. According to (2.8), (2.9), (2.10) and (2.18), the first step of the state transformation prcess is obtained with the transformation

$$\boldsymbol{x} = \begin{bmatrix} \mathbf{I} & \boldsymbol{0} \\ \boldsymbol{p}_n^T & 1 \end{bmatrix} \boldsymbol{y}_n \quad , \tag{2.24}$$

with $\boldsymbol{p}_n^T \in \mathbb{R}^{1\times(n-1)}$.
The substitution of (2.24) into (2.2) yields

$$\dot{\boldsymbol{y}}_n = \begin{bmatrix} \mathbf{I}_{n-1}^+ + \boldsymbol{e}_{n-1}\boldsymbol{p}_n^T & \boldsymbol{e}_{n-1} \\ -\boldsymbol{p}_n^T\mathbf{I}_{n-1}^+ - \boldsymbol{p}_n^T\boldsymbol{e}_{n-1}\boldsymbol{p}_n^T + \boldsymbol{q}_n^T + \tilde{q}_n\boldsymbol{p}_n^T - \dot{\boldsymbol{p}}_n^T & -\boldsymbol{p}_n^T\boldsymbol{e}_{n-1} + \tilde{q}_n \end{bmatrix} \boldsymbol{y}_n \quad . \tag{2.25}$$

Define

$$\lambda_n = \tilde{q}_n - p_{n,n-1} \quad . \tag{2.26}$$

Moreover we demand

$$\dot{p}_{n,i} = p_{n,i-1} + q_i + \lambda_n p_{n,i} \quad (i = 1, 2, \ldots, n-1) \tag{2.27}$$

for the components $p_{n,i}$ of $\boldsymbol{p}_n^T$.
And (2.25) becomes

$$\dot{\boldsymbol{y}}_n = \begin{bmatrix} \mathbf{I}_{n-1}^+ + \boldsymbol{e}_{n-1}\boldsymbol{p}_n^T & \boldsymbol{e}_{n-1} \\ \boldsymbol{0}^T & \lambda_n \end{bmatrix} \boldsymbol{y}_n \quad . \tag{2.28}$$

Here

$$\mathbf{I}_{n-1}^+ + \boldsymbol{e}_{n-1}\boldsymbol{p}_n^T = \begin{bmatrix} \mathbf{I}_{n-2}^+ & \boldsymbol{e}_{n-2} \\ \boldsymbol{q}_{n-1}^T & \tilde{q}_{n-1} \end{bmatrix} \quad , \tag{2.29}$$

where

$$\boldsymbol{q}_{n-1}^T(t) = [\, p_{n,1} \ \ldots \ p_{n,n-2} \,] \quad , \tag{2.30}$$

$$\tilde{q}_{n-1}(t) = p_{n,n-1} \quad . \tag{2.31}$$

The comparison between the matrices (2.2) and (2.29) will show that for this scalar system the iterative procedure becomes simply a repetition with matrices of smaller dimensions. The complete calculation scheme for the results of the scalar system are summarized in Figure 1.

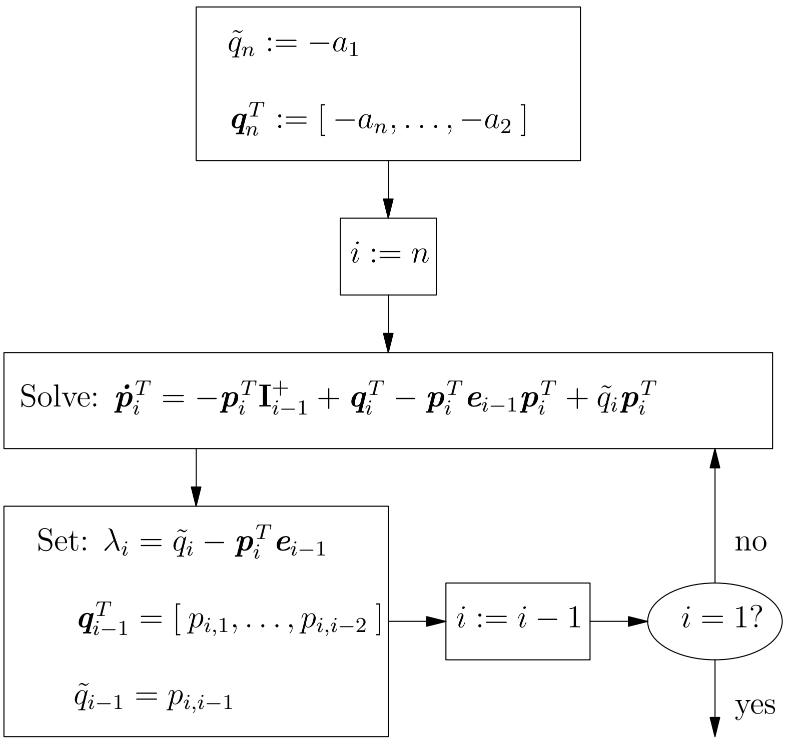

Figure 1: Algorithm

It is noticed that Zhu [4] introduced two different sets $\{\lambda_i(t); i = 1, \ldots, n\}$ and $\{\rho(t); i = 1, \ldots, n\}$ of dynamic eigenvalues. The first set coincide with the dynamic eigenvalues introduced here. The second set is associated with a constant amplitude representation of each mode. Thus, the elementary mode

$$u_i(t)\exp[\int_0^t \lambda_i(\tau)d\tau] \tag{2.32}$$

is written as

$$c_i \exp[\int_0^t \rho_i(\tau)d\tau] \tag{2.33}$$

where [5]

$$\rho_i(t) = \lambda_i(t) + \dot{u}_i(t)u_i^{-1}(t) \quad . \tag{2.34}$$

It is noted that for more-dimensional systems, difficulties will arise with expression (2.34), since $u_i(t)$ as well as c_i are then vectors.

3 Characteristic equations

In section 2 a number of transformation matrices are introduced such that the system matrix is transformed into $\Lambda(t) + \mathbf{I}^+$ in which $\Lambda(t)$ is a diagonal matrix and $\mathbf{I}^+$, is defined in (2.5). This is done in a successive procedure and each step asks for the solution of a Riccati equation (2.21). The dynamic eigenvalue is then obtained in a algebraic manner.
In each step (2.21) and (2.22), there are $i-1$ differential equations for $\boldsymbol{p}_i^T$ and 1 algebraic equation for $\lambda_i(i = n, \ldots, 2)$. So the number of equation and unknowns is equal. As a consequence, an elimination process can be designed such that a differential equation for $\lambda_i(i = n, \ldots, 2)$ is obtained. For time invariant systems the left hand side of (2.25) equals zero and the elimination proces yield the well-known characteristic equations [9]. In [3] and [10] such elimination processes have been performed for linear time-varying systems. For a third-order system ($n = 3$) the equations (2.21) and (2.22) become for $i = 3$

$$\dot{p}_{31} = +\lambda_3 p_{31} - a_3 \quad , \quad \dot{p}_{32} = +\lambda_3 p_{32} - p_{31} - a_2 \quad , \quad \lambda_3 = -p_{32} - a_1 \tag{3.1}$$

Remark the difference in notation of (3.1) with (3.6) and (3.4) in [10]. For the same third order system (2.21) and (2.22) yield for $i = 2$

$$\dot{p}_{21} = \lambda_2 p_{21} + p_{32} \quad , \quad \lambda_2 = p_{32} - p_{21} \tag{3.2}$$

The trace preserving property of the Riccati transformation as described with the matrices (2.16) and (2.18) yields

$$\lambda_1 + \lambda_2 + \lambda_3 = -a_1 \tag{3.3}$$

As is shown in [10] the equations (3.1) (3.2) and (3.3) yield

$$\ddot{\lambda}_1 + 3\lambda_1\dot{\lambda}_1 + \lambda_1^3 + a_1(\lambda_1^2 + \dot{\lambda}_1) + a_2\lambda_1 + a_3 = 0 \tag{3.4}$$

$$\dot{\lambda}_2 + \lambda_2^2 + \lambda_2\lambda_1 + a_1(\lambda_2 + \lambda_1) + \lambda_1^2 + 2\dot{\lambda}_1 + a_2 = 0 \tag{3.5}$$

$$\lambda_3 + \lambda_2 + \lambda_1 = -a_1 \tag{3.6}$$

It is seen, from (3.1)-(3.3) as well from (3.4)-(3.6) that for LTV systems the solution of the dynamic eigenvalue has to be performed in a specific order. Remark that for LTI the derivatives in (3.4) disappear and then the classical characteristic equation appears.

4 An example: the third order Euler equation

Consider as an example the third order Euler differential equation [11]

$$(t^3D^3 - t^2D^2 - 2tD - 4)x = 0 \tag{4.1}$$

or its equivalent state-space description

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4t^{-3} & 2t^{-2} & t^{-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad . \tag{4.2}$$

For this system the equations (3.1), (3.2), (3.3) will give as solutions

$$p_{31} = -t^{-2},\ p_{32} = -t^{-1},\ \lambda_3 = 2t^{-1} \quad , \tag{4.3}$$

$$p_{21} = jt^{-1},\ \lambda_2 = -(1+j)t^{-1} \quad , \tag{4.4}$$

$$\lambda_1 = jt^{-1} \quad . \tag{4.5}$$

Remark that the (complex) dynamic eigenvalues $\lambda_1(t)$ and $\lambda_2(t)$ in general are *not* complex conjugated:

$$\mathrm{Im}\lambda_1 = -\mathrm{Im}\lambda_2 \quad \text{while} \quad \mathrm{Re}\lambda_1 \neq \mathrm{Re}\lambda_2 \quad . \tag{4.6}$$

The values obtained for λ_1 and λ_2 also satisfy (3.4), (3.5) and (3.6).

References

[1] B. van der Pol, "The Fundamental Principles of Frequency Modulation," *Journ. I.E.E.*, Vol. 93, Part III, No. 23, May, 1946, pp. 1126-1131.

[2] P. van der Kloet and F.L. Neerhoff, "Modal Factorization of Time-Varying Models for Nonlinear Circuits by the Riccati Transform," *Proc. IEEE International Symposium on Circuits and Systems (ISCAS) 2001*, Sydney, Australia, May 2001, pp. III-553-556.

[3] E.W. Kamen, "The Poles and Zeros of a Linear Time-Varying System," *Lin. Alg. and its Appl.*, Vol. 98, 1988, pp. 273-289.

[4] J. Zhu and C.D. Johnson, "Unified Canonical Forms for Matrices Over a Differential Ring," *Lin. Algebra and its Appl.*, Vol. 147, March 1991, pp. 201-248.

[5] P. van der Kloet and F.L. Neerhoff, "Normalized Dynamic Eigenvalues for Scalar Time-Varying Systems," *Proc. Nonlinear Dynamics of Electronic Systems (NDES) 2002*, Izmir, Turkey, 21-23 June, 2002.

[6] E.L. Ince, "Ordinary Differential Equations," *Dover Publ. Inc.*, New York (first edition 1926), 1956, pp. 121.

[7] A.C. Amitsur, "Differential Polynomials and Division Algebras," *Ann. Math.*, Vol. 59, 1954, pp. 245-278.

[8] D.R. Smith, "Decoupling and Order Reduction via the Riccati Transformation," *SIAM Review*, Vol. 29, No. 1, March 1987, pp. 91-113.

[9] P. van der Kloet and F.L. Neerhoff, "The Riccati Equation as Characteristic Equation for General Linear Dynamic Systems," *Nonlinear Theory and its Applications (NOLTA) 2001*, Miyagi, Zao, Japan, 28 Oct - 1 Nov, 2001, Vol. 2, pp. 425-428.

[10] F.L. Neerhoff and P. van der Kloet, "The Characteristic Equation for Time-Varying Models of Nonlinear Dynamic Systems," *Proc. European Conference on Circuit Theory and Design (ECCTD) 2001*, Espoo, Finland, August 28-31, 2001, pp. III-125-128.

[11] E.L. Ince, "Ordinary Differential Equations," *Dover Publ. Inc.*, New York (first edition 1926), 1956, pp. 141.