NORMALIZED DYNAMIC EIGENVALUES FOR SCALAR TIME-VARYING SYTEMS

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Abstract— *Linear time-varying systems are considered. The associated homogeneous time-varying differential equation is assumed to be given in a frame of reference such that the system matrix is upper triangular. An analytic expression for the solution then can be derived. For a higher order SISO system this solution is a sum of modes, each mode being the product of constant amplitude and an exponential function whose argument contains the normalized dynamic eigenvalues.*

I. INTRODUCTION

In this paper single input single output (SISO), together with multiple input multiple output (MIMO) linear time-varying (LTV) systems will be discussed. For these systems the fundamental solution in state-space can be written as the product of two matrices, one containing the so-called dynamic eigenvectors, while the other matrix, a diagonal one, contains the dynamic eigenvalues. This is established in the accompanion paper [1]. The structure of the fundamental matrix yields that every solution of the homogeneous equation is the sum of elementary modes, each elementary mode being the product of a time-varying amplitude and an exponential function.

For SISO systems, Kamen [2] and Zhu [3] have given representation of the solution in which the elementary modes have a constant amplitude, while their exponential functions contain a type of eigenvalue which is called right pole by Kamen and PD-eigenvalue by Zhu.

As a consequence, at least for SISO systems, there are defined two different sets of eigenvalues. The uniqueness theorem states the unicity of the solutions; so they must be related. It is proved here that the two sets are dynamic similar.

For that purpose first the state space equation for MIMO system is transformed such that the system matrix becomes upper triangular. This needs a lower triangular matrix for the Riccati transformation. The solution of the differential equation associated with this triangular system matrix can be obtained in analytical form. For this triangular system the fundamental solution will be obtained, also in the form of a triangular matrix. The elements of the fundamental matrix will be expressed in the elements of the system matrix. This will be applied to a SISO system for which specific values can be given to the coefficients of the triangular system matrix. The solution for the SISO LTV system can be written in a representation with constant amplitudes. The exponential functions in the solution then will have other arguments than with the dynamic eigenvalues. This change will define what we call the normalized dynamic eigenvalues. The comparison with litarature shows that they are equivalent to the right poles of Kamen [2] and the PD-eigenvalues of Zhu [3]. It will appear that for SISO LTV systems the different sets of eigenvalues are dynamical similar.

The approach for MIMO systems goes back to the work of Wu [4], [5]. He introduced the concept of a dynamic eigenpair. Such a dynamic eigenpair is a combination of a dynamic eigenvalue and a dynamic eigenvector. Apart from the homogeneous differential equation that such a combination has to satisfy a second differential equation for the dynamic eigenvectors. For linear time invariant systems with constant eigenvectors this second differential equation reduces to the characteritic equation.

II. LINEAR TIME VARYING SYSTEMS

Consider the linear time-varying system

$$\dot{\boldsymbol{x}} = \mathbf{A}(t)\boldsymbol{x} \quad , \tag{1}$$

with $\boldsymbol{x} \in \mathbb{R}^n$ and $\mathbf{A} \in \mathbb{R}^{n\times n}$. In [1] it is shown that there exist a compound Riccati transformation [6]

$$\boldsymbol{x} = \mathbf{R}(t)\boldsymbol{y} \quad , \tag{2}$$

such that $\boldsymbol{y}$ satisfies

$$\dot{\boldsymbol{y}} = \mathbf{B}(t)\boldsymbol{y} \quad , \tag{3}$$

with $\mathbf{B} \in \mathbb{R}^{n\times n}$ an upper triangle matrix.
The diagonal elements λ_i $(i = 1, 2, \dots, n)$ are dynamic eigenvalues of $\mathbf{B}$ and also of $\mathbf{A}$, which is dynamic similar with $\mathbf{B}$.
Here we state that the funadamental matrix $\boldsymbol{\Phi}(t, 0)$ of (3) can be written as

$$\boldsymbol{\Phi}(t, 0) = \mathbf{F}(t) \exp \boldsymbol{\Gamma}(t) \quad (4)$$

with

$$\boldsymbol{\Gamma}(t) = \int_0^t \boldsymbol{\Lambda}(\tau) d\tau \;,\; \boldsymbol{\Lambda}(t) = \mathrm{diag}(\lambda_1, \dots, \lambda_n) \quad (5)$$

We must have

$$\mathbf{F}(0) = \mathbf{I} \quad (6)$$

otherwise $\boldsymbol{\Phi}(t, 0)$ will not represent the fundamental sollution of (3). The substitution into (3) of

$$\boldsymbol{y}(t) = \mathbf{F}(t) \exp\{\boldsymbol{\Gamma}(t)\} \boldsymbol{y}(0) \quad (7)$$

with $\boldsymbol{\Gamma}(t)$ according to (5) will yield

$$(\dot{\mathbf{F}} + \mathbf{F}\boldsymbol{\Lambda} - \mathbf{B}\mathbf{F}) \exp \boldsymbol{\Gamma}(t) \boldsymbol{y}(0) = \boldsymbol{0} \quad (8)$$

This holds for all initial values $\boldsymbol{y}(0)$ if $\mathbf{F}$ satisfies the matrix differential equation [7]

$$\dot{\mathbf{F}} + \mathbf{F}\boldsymbol{\Lambda} - \mathbf{B}\mathbf{F} = \mathbf{O} \quad (9)$$

In (9) the marices $\mathbf{B}$ and $\boldsymbol{\Lambda}$ are dynamical similar, because they share the dynamic eigenvalues $\lambda_i(t)$.
If we first solve the last row of equation in (9) we will find

$$\dot{f}_{nj} - (\lambda_n - \lambda_j) f_{nj} = 0 \quad (10)$$

Thus with (6) it follows

$$\left.\begin{aligned} &f_{nj} = 0 \;\; (j = 1, 2, \dots, n-1) \\ &f_{nn} = 1 \end{aligned}\right\} \quad (11)$$

In the next steps $f_{n-1,1}, \dots, f_{n-1,n-2}, f_{n-1,n-1}$, $f_{n-2,1}, \dots, f_{n-2,n-3}, f_{n-2,n-2}, \dots, f_{3,1}, f_{3,2}, f_{3,3}$, $f_{2,1}, f_{2,2}, f_{1,1}$ will be solved from (10), yielding with the help of (6)

$$\left.\begin{aligned} &f_{ij} = 0 \;\; (i > j) \\ &f_{ii} = 1 \end{aligned}\right\} \quad (12)$$

For $j = i + l$ $(l = 1, 2, \dots, n - i)$ (9) yields

$$\dot{f}_{ij} = (\lambda_i - \lambda_j) f_{ij} + \sum_{l=i+1}^{j} b_{i,l} f_{l,j} \quad (13)$$

We will here state the solutions

$$f_{i,i+1}(t) = \phi^{-1}_{i+1,i}(t) \int_0^t \phi_{i+1,i}(\tau) b_{i,i+1}(\tau) d\tau \quad (14)$$

for (i=1, . . . , n-1) and

$$\begin{aligned} f_{i,i+2}(t) = \;& \phi^{-1}_{i+2,i}(t) \int_0^t \phi_{i+2,i}(\tau) b_{i,i+2}(\tau) d\tau + \\ &+ \phi^{-1}_{i+2,i+1}(t) \int_0^t \Big\{ \phi_{i+1,i}(\tau_1) \times \\ &\times \int_0^{\tau_1} \phi_{i+2,i+1}(\tau_2) b_{i+1,i+2}(\tau_2) d\tau_2 \Big\} d\tau_1 \end{aligned} \quad (15)$$

for (i=1, . . . , n-2) where

$$\phi_{ij}(t) = \exp\{\gamma_i(t) - \gamma_j(t)\} \quad (16)$$

The other solutions of (13) have a similar structure.

III. THE SISO LTV SYSTEM

In this section the general solution of

$$D^n x + a_1(t) D^{n-1} x + \dots + a_{n-1}(t) D x + a_n(t) x = 0 \quad (17)$$

will be derived in terms of dynamic eigenvalues. Here D denotes the time derative d/dt. For that purpose (17) is first formulated in state space

$$\dot{\boldsymbol{x}} = \mathbf{A}_c(t) \boldsymbol{x} \quad (18)$$

where $\mathbf{A}_c(t) \in \mathbb{R}^{n\times n}$ is the Frobenius companion matrix and $\boldsymbol{x} \in \mathbb{R}^n$ the state vector whose first component x_1 equals the unknown function x in (17)

$$x(t) = x_1(t) \quad (19)$$

The equation (18) can be transformed to an upper diagonal representation, as described in [9], such that we obtain

$$\dot{\boldsymbol{y}} = \mathbf{B}(t) \boldsymbol{y} \quad (20)$$

with $\mathbf{B}$ an upper triangular matrix defined by

$$\left.\begin{aligned} b_{ii} &= \lambda_i(t) \; (i = 1, 2, \dots, n) \\ b_{ii+1} &= 1 \; (i = 1, 2, \dots, n-1) \\ b_{ii+l} &= 0 \; (l = 2, 3, \dots, n-i) \end{aligned}\right\} \quad (21)$$

and

$$x(t) = y_1(t) \quad (22)$$

where $y_1(t)$ is the first component of $\boldsymbol{y}$.
Remark that $\lambda_i(t)$ is a dynamic eigenvalue of $\mathbf{A}_c$. The solution proposed in the preceding section

$$\boldsymbol{y}(t) = \mathbf{F}(t) \exp\{\boldsymbol{\Gamma}(t)\} \boldsymbol{y}(0) \quad (23)$$

for (22) will yield here the eigenvalue problem

$$\dot{\mathbf{F}} = (\mathbf{\Lambda} + \mathbf{I}_n^+)\mathbf{F} - \mathbf{F}\mathbf{\Lambda} \tag{24}$$

Here $\mathbf{I}_n^+$ is the diagonal shift matrix

$$\mathbf{I}_n^+ = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \dots & \dots & 0 \end{bmatrix} \tag{25}$$

We find for (24), in agreement with (13)

$$\dot{f}_{ij} = (\lambda_i - \lambda_j) f_{ij} + f_{i+1,j} \tag{26}$$

Remark that (22) and (23) yield

$$x(t) = y_1(t) = \sum_{j=1}^{n} f_{1j}(t) \exp\{\gamma_j(t)\} y_j(0) \tag{27}$$

Here

$$\gamma_j(t) = \int_0^t \lambda_j(\tau) d\tau \tag{28}$$

To find $f_{1j}(t)$ we could explore (14) and (15) for suitable chosen functions $b_{ij}(t)$, but we can also solve (26) directly

$$f_{ij}(t) = \int_0^t \exp[\{\gamma_i(t) - \gamma_j(t)\} - \{\gamma_i(\tau) - \gamma_j(\tau)\}] f_{i+1,j}(\tau) d\tau \tag{29}$$

This yields directly

$$f_{i,i+1}(t) = \phi_{i+1,i}^{-1}(t) \int_0^t \phi_{i+1,i}(\tau) d\tau \tag{30}$$

with

$$\phi_{ij}(t) = \exp\{\gamma_i(t) - \gamma_j(t)\} \tag{31}$$

Moreover

$$f_{i,i+2}(t) = \phi_{i+2,i}^{-1}(t) \int_0^t \phi_{i+2,i+1}(\tau_1) \int_0^{\tau_1} \phi_{i+1,i}(\tau_2) d\tau_2 d\tau_1 \tag{32}$$

and finally

$$f_{ij}(t) = \phi_{j,i}^{-1}(t) \int_0^t \phi_{j,j-1}(\tau_1) \int_0^{\tau_1} \phi_{j-1,j-2}(\tau_2) \dots \dots \int_0^{\tau_{j-i-1}} \phi_{i+1,i}(\tau_{j-i}) \dots \dots d\tau_{j-i} \dots d\tau_2 d\tau_1 \tag{33}$$

Particularly, we have to substitute

$$f_{1j}(t) = \phi_{j,1}^{-1}(t) \int_0^t \phi_{j,j-1}(\tau_1) \int_0^{\tau_1} \phi_{j-1,j-2}(\tau_2) \dots \int_0^{\tau_{j-2}} \phi_{2,1}(\tau_{j-1}) d\tau_{j-1} \dots d\tau_2 d\tau_1 \tag{34}$$

into (27) to obtain the solytion of (17). We will write this solution of (17) as

$$x(t) = \sum_{j=1}^{n} g_j(t) \exp\{\gamma_1(t)\} y_j(0) \tag{35}$$

where

$$g_j(t) = \phi_{j1}(t) f_{1j}(t) \tag{36}$$

IV. NORMALIZED DYNAMIC EIGENVALUES

The solution (35) of (17) can also be written as

$$x(t) = \sum_{j=1}^{n} \exp\{\int_0^t \rho_j(\tau) d\tau\} y_j(0) \tag{37}$$

where

$$\rho_j(t) = \lambda_1(t) + \frac{1}{g_j(t)} \dot{g}_j(t) \tag{38}$$

These functions $\rho_j(t)$ $(j = 1, 2, \dots, n)$ are called normalized dynamic eigenvalues and coincide with the PD-eigenvalues of Zhu [3].
With (36) and (38), there follows

$$\rho_j(t) = \lambda_1(t) + \frac{\dot{\phi}_{j1}}{\phi_{j1}} + \frac{\dot{f}_{1j}}{f_{1j}} \tag{39}$$

and in view of (31) thus

$$\rho_j(t) = \lambda_j(t) + \frac{\dot{f}_{1j}}{f_{1j}} \tag{40}$$

So

$$\rho_j(t) = f_{1j}(t) \lambda_j(t) f_{1j}^{-1}(t) - f_{1j}(t) (f_{1j}^{-1}(t))^{\cdot} \tag{41}$$

We can write (41) in matrix form

$$\mathbf{P}(t) = \mathbf{D}^{-1}(t) \mathbf{\Lambda}(t) \mathbf{D}(t) - \mathbf{D}^{-1}(t) \dot{\mathbf{D}}(t) \tag{42}$$

where all matrices are diagonal

$$\mathbf{D}(t) = \text{diag}\{f_{11}^{-1}(t), \dots, f_{1n}^{-1}(t)\} \tag{43}$$

$$\mathbf{P}(t) = \text{diag}\{\rho_1(t), \dots, \rho_n(t)\} \tag{44}$$

and $\mathbf{\Lambda}(t)$ as defined in (5). The meaning of (42) is that the two sets $\{\rho_1, \ldots, \rho_n\}$ and $\{\lambda_1, \ldots, \lambda_n\}$ are dynamical equivalent. So the differential equation

$$\dot{\boldsymbol{z}} = \mathbf{\Lambda} \boldsymbol{z} \tag{45}$$

will be transformed into

$$\dot{\boldsymbol{u}} = \mathbf{P} \boldsymbol{u} \tag{46}$$

under the state transformation

$$\boldsymbol{z} = \mathbf{D} \boldsymbol{u} \tag{47}$$

It will be clear that the compound state transformation

$$\boldsymbol{x}(t) = \mathbf{R}(t)\mathbf{F}(t)\mathbf{D}(t)\boldsymbol{u}(t) \tag{48}$$

transforms the original equation (18) into(46). The solution of (18) thus can be written as

$$\boldsymbol{x}(t) = \mathbf{R}(t)\mathbf{F}(t)\mathbf{D}(t) \exp\{\int_0^t \mathbf{P}(\tau)d\tau\}\boldsymbol{u}(0) \tag{49}$$

Here

$$\exp\Big[\int_0^t \mathbf{P}(\tau)d\tau\Big] = \text{diag}\Big[\exp\{\int_0^t \rho_1(\tau)d\tau\}, \ldots, \exp\{\int_0^t \rho_n(\tau)d\tau\}\Big] \tag{50}$$

V. CONCLUSIONS

During the last 20 years a number of attempts has been done to generalize the eigenvalue problem for linear time varying systems. It is shown here that the dynamic eigenvalues and eigenvectors as introduced in the accompanion paper hold for multiple input multiple output systems (LTV and LTI). The reduction to SISO systems show that they are then dynamic similar to earlier introduced sets of normalized eigenvalues, which are not defined for MIMO systems.

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