THE CAUCHY-FLOQUET FACTORIZATION BY SUCCESSIVE RICCATI TRANSFORMATIONS

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ABSTRACT

Scalar linear time-varying systems are addressed. In particular, a new factorization method for the associated scalar polynomial system differential operator is presented. It differs from the classical results due to Cauchy and Floquet, in that it is based upon successive Riccati transformations of the Frobenius companion system matrix. As a consequence, the factorization is obtained in terms of the earlier introduced dynamic eigenvalues.

1. INTRODUCTION

In this paper, linear time-varying (LTV) single-input single-output (SISO) systems are addressed. In particular, a new factorization method for the associated scalar polynomial system differential operator is presented. As such, it is an alternative method in obtaining the classical results due to Cauchy with respect to constant systems [1] and Floquet for varying systems [2,3], respectively.

Our approach is based upon successive applications of the Riccati transformation as described in [4]. As a result, the Frobenius companion system matrix is gradually triangularized with the earlier introduced dynamic eigenvalues on its main diagonal [5,6,7,8]. Finally, the original system differential operator is factorized in terms of the dynamic eigenvalues. Since the classical Cauchy-Floquet factorization is the basis of an eigenvalue theory developed by Zhu *et.al.* [9,10,11,12], it follows that their so-called series D-eigenvalues are identical to our dynamic eigenvalues.

In the next section, first the terminology with respect to general LTV systems is shortly recapitulated. Then, in Section 3 the system matrix is gradually triangularized by successive Riccati transformations. Finally, in Section 4 scalar LTV systems are addressed. In particular, it is shown that the associated polynomial system differential operator, namely

$$L = D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n \quad , \tag{1}$$

in which $D = d/dt$ and $a_i = a_i(t)$ $(i = 1, \ldots, n)$ are the time-dependent system parameters, respectively, is factorized as

$$L = (D - \lambda_n)(D - \lambda_{n-1}) \ldots (D - \lambda_1) \quad , \tag{2}$$

where $\lambda_i = \lambda_i(t)$ $(i = 1, 2, \ldots, n)$ denote the dynamic eigenvalues of the scalar LTV system under consideration.

2. GENERAL LTV SYSTEMS

First, consider the n-dimensional homogeneous LTV system

$$\dot{\boldsymbol{x}} = \mathbf{A}(t)\boldsymbol{x} \quad , \tag{3}$$

in which $\boldsymbol{x} = \boldsymbol{x}(t)$ denotes an unknown n-dimensional column-vector and $\mathbf{A}$ the known n-dimensional system matrix, respectively. We are looking for elementary solutions of the modal form [6,7,8]

$$\boldsymbol{x}(t) = \boldsymbol{u}(t) \exp[\gamma(t)] \quad , \tag{4}$$

where $\boldsymbol{u}$ denotes a varying amplitude-vector, while the varying phase γ defines a varying frequency λ as

$$\lambda(t) = \dot{\gamma}(t) \quad \text{with} \quad \gamma(t) = \int_0^t \lambda(\tau) d\tau \quad . \tag{5}$$

Substitution of (4) in the state-equation (3) yields the *dynamic* eigenvalue problem [6,7,8]

$$[\mathbf{A}(t) - \lambda(t)\mathbf{I}]\boldsymbol{u}(t) = \dot{\boldsymbol{u}}(t) \quad , \tag{6}$$

in which $\mathbf{I}$ denotes the identity matrix. A solution of (3) which *also* satisfies (6) will be called an elementary mode of (3). In this context, the modal quantities $\boldsymbol{u}$ and λ are called a *dynamic* eigenvector and a *dynamic* eigenvalue, respectively. In order to solve (6) for $\boldsymbol{u}$ and λ, system (3) is subjected to the time-dependent coordinate transformation

$$\boldsymbol{x} = \mathbf{R}(t)\boldsymbol{y} \quad , \tag{7}$$

where $\boldsymbol{y} = \boldsymbol{y}(t)$ is the new unknown. Then, system (3) goes into another LTV system, namely

$$\dot{\boldsymbol{y}} = \mathbf{B}(t)\boldsymbol{y} \quad , \tag{8}$$

in which the system matrix $\mathbf{B}$ is given by [13]

$$\mathbf{B} = \mathbf{R}^{-1}\mathbf{A}\mathbf{R} - \mathbf{R}^{-1}\dot{\mathbf{R}} \quad . \tag{9}$$

It is easily shown that system (3) and (8) share the same dynamic eigenvalues. Therefore, the system matrices $\mathbf{A}$ and $\mathbf{B}$ are called *dynamically* similar [6].In the next section, we construct a coordinate transformation matrix $\mathbf{R}$ by which system (3) is gradually triangularized. As shown earlier, in each triangularization step, a next dynamic eigenvalue appears on the main diagonal. Once the system is fully triangularized, the complete dynamic eigenvalue-spectrum is known. Finally, the associated dynamic eigenvectors are easily found by back substitution and straightforward integration [8].

3. TRIANGULARIZATION BY THE RICCATI TRANSFORMATION

In this section, we develop an algorithm by which (3) is triangularized as

$$\dot{\boldsymbol{y}} = \mathbf{B}(t)\boldsymbol{y} \tag{10}$$

with $\mathbf{B}$ uppertriangular. Here, $\mathbf{B}$ is iteratively obtained as $(k = n, \ldots, 2)$

$$\left.\begin{aligned} \mathbf{A}_n &= \mathbf{A} \\ \mathbf{A}_{k-1} &= \mathbf{R}_k^{-1}\mathbf{A}_k\mathbf{R}_k - \mathbf{R}_k^{-1}\dot{\mathbf{R}}_k \\ \mathbf{B} &= \mathbf{A}_1 \end{aligned}\right\} \tag{11}$$

corresponding with a successive Riccati transformation

$$\boldsymbol{x}(t) = \mathbf{R}_n(t)\mathbf{R}_{n-1}(t)\ldots\mathbf{R}_2(t)\boldsymbol{y}(t) \tag{12}$$

of system (3) into system (10).
Now assume that $\mathbf{A}_k(t)$ has the following partitionized form

$$\mathbf{A}_k(t) = \begin{bmatrix} \mathbf{A}_{11}^{(k)}(t) & \mathbf{A}_{12}^{(k)}(t) \\ \mathbf{0} & \mathbf{A}_{22}^{(k)}(t) \end{bmatrix} \quad , \tag{13}$$

with $\mathbf{A}_{11}^{(k)}(t)$ and $\mathbf{A}_{22}^{(k)}(t)$ square matrices and $\mathbf{A}_{12}^{(k)}(t)$ a $k \times (n-k)$ matrix. Moreover, assume that $\mathbf{A}_{22}^{(k)}(t)$ is triangular. Assume further

$$\mathbf{R}_k(t) = \begin{bmatrix} \mathbf{P}_k(t) & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-k} \end{bmatrix} \tag{14}$$

with the same partitioning as $\mathbf{A}_k(t)$. Now, the general step in (11) yields with (13) and (14)

$$\mathbf{A}_{k-1}(t) = \begin{bmatrix} \mathbf{P}_k^{-1}\mathbf{A}_{11}^{(k)}(t)\mathbf{P}_k - \mathbf{P}_k^{-1}\dot{\mathbf{P}}_k & \mathbf{P}_k^{-1}\mathbf{A}_{12}^{(k)}(t) \\ \mathbf{0} & \mathbf{A}_{22}^{(k)}(t) \end{bmatrix} \tag{15}$$

Next, the partitionings (T stands for the transpose)

$$\mathbf{A}_{11}^{(k)} = \begin{bmatrix} \mathbf{D}_k & \boldsymbol{b}_k \\ \boldsymbol{c}_k^T & d_{kk} \end{bmatrix} \quad \text{and} \quad \mathbf{A}_{12}^{(k)} = \begin{bmatrix} \tilde{\mathbf{D}}_k \\ \tilde{\boldsymbol{c}}_k^T \end{bmatrix} \tag{16}$$

are used to show that for

$$\mathbf{P}_k(t) = \begin{bmatrix} \mathbf{I}_{k-1} & \boldsymbol{0} \\ \boldsymbol{p}_k^T & 1 \end{bmatrix} \tag{17}$$

with $\boldsymbol{p}_k^T = [\, p_{k1} \ldots p_{k,k-1} \,]$ denotes any solution of the differential Riccati equation

$$\dot{\boldsymbol{p}}_k^T = -\boldsymbol{p}_k^T\mathbf{D}_k + \boldsymbol{c}_k^T - \boldsymbol{p}_k^T\boldsymbol{b}_k\boldsymbol{p}_k^T + d_{kk}\boldsymbol{p}_k^T \quad , \tag{18}$$

equation (15) can be written as

$$\begin{aligned} &\mathbf{A}_{k-1} = \\ &= \begin{bmatrix} \begin{bmatrix} \mathbf{D}_k + \boldsymbol{b}_k\boldsymbol{p}_k^T & \boldsymbol{b}_k \\ \boldsymbol{0}^T & -\boldsymbol{p}_k^T\boldsymbol{b}_k + d_{kk} \end{bmatrix} & \begin{bmatrix} \tilde{\mathbf{D}}_k \\ -\boldsymbol{p}_k^T\tilde{\mathbf{D}}_k + \tilde{\boldsymbol{c}}_k^T \end{bmatrix} \\ \mathbf{0} & \mathbf{A}_{22}^{(k)} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{D}_k + \boldsymbol{b}_k\boldsymbol{p}_k^T & \begin{bmatrix} \boldsymbol{b}_k & \tilde{\mathbf{D}}_k \end{bmatrix} \\ \begin{bmatrix} \boldsymbol{0}^T \\ \mathbf{0} \end{bmatrix} & \begin{bmatrix} -\boldsymbol{p}_k^T\boldsymbol{b}_k + d_{kk} & -\boldsymbol{p}_k^T\tilde{\mathbf{D}}_k + \tilde{\boldsymbol{c}}_k^T \\ \boldsymbol{0} & \mathbf{A}_{22}^{(k)} \end{bmatrix} \end{bmatrix} \end{aligned} \tag{19}$$

As a result, we arrive at the partitioning (13) for $\mathbf{A}_{k-1}$ with

$$\left.\begin{aligned} \mathbf{A}_{11}^{(k-1)} &= \mathbf{D}_k + \boldsymbol{b}_k\boldsymbol{p}_k^T \\ \mathbf{A}_{12}^{(k-1)} &= \begin{bmatrix} \boldsymbol{b}_k & \tilde{\mathbf{D}}_k \end{bmatrix} \\ \mathbf{A}_{22}^{(k-1)} &= \begin{bmatrix} -\boldsymbol{p}_k^T\boldsymbol{b}_k + d_{kk} & -\boldsymbol{p}_k^T\tilde{\mathbf{D}}_k + \tilde{\boldsymbol{c}}_k^T \\ \boldsymbol{0} & \mathbf{A}_{22}^{(k)} \end{bmatrix} \end{aligned}\right\} \tag{20}$$

We finally conclude that $\mathbf{A}_{22}^{(k-1)}$ is triangular if $\boldsymbol{p}_k^T$ satisfies (18) and if $\mathbf{A}_{22}^{(k)}$ is triangular.
The Riccati transfomations $\mathbf{R}_k$ each satisfies

$$\det \mathbf{R}_k = 1 \quad \Rightarrow \quad \det \mathbf{R} = \det(\mathbf{R}_n \ldots \mathbf{R}_2) = 1 \tag{21}$$

As a consequence, the Riccati transformation $\mathbf{R}$ is trace preserving, thus

$$\text{trace}\,[\mathbf{A}(t)] = \text{trace}\,[\mathbf{B}(t)] = \sum_{i=1}^{n} \lambda_i(t) \tag{22}$$

with $\lambda_i(t)$ the elements on the main diagonal of $\mathbf{B}$. These functions $\lambda_i(t)$ are the dynamic eigenvalues of $\mathbf{B}$ [7, 8].

4. SCALAR LTV SYSTEMS

We now turn to scalar LTV systems, characterized by the n-th order linear inhomogeneous differential equation with normalized time-dependent coefficients $a_i = a_i(t)$ $(i = 1, 2, \ldots, n)$ [9,10,16]

$$D^n x_1 + a_1 D^{n-1} x_1 + \cdots + a_{n-1} D x_1 + a_n x_1 = f \quad , \tag{23}$$

where $f = f(t)$ and $x_1 = x_1(t)$ are the input and the output variable, respectively, while $D = d/dt$. By introducing the new variables $\{x_2, x_3, \ldots, x_n\}$ as

$$\left.\begin{aligned} x_2 &= \dot{x}_1 \\ x_3 &= \dot{x}_2 \\ &\ldots \\ x_n &= \dot{x}_{n-1} \end{aligned}\right\} \tag{24}$$

the scalar equation (23) is recasted in state-space description with the Frobenius companion matrix $\mathbf{A} = \mathbf{A}_n$, given by

$$\mathbf{A}_n(t) = \begin{bmatrix} \mathbf{I}^+_{n-1} & \boldsymbol{e}_{n-1} \\ \boldsymbol{q}^T_n & q_{n,n} \end{bmatrix} \tag{25}$$

in which

$$\left.\begin{aligned} \boldsymbol{q}^T_n &= -\,[a_n \, \ldots , a_2\,] \\ q_{n,n} &= -\,a_1 \end{aligned}\right\} \tag{26}$$

while $\mathbf{I}^+_k$ denotes the square shift matrix of size k, given by

$$\mathbf{I}^+_k = \begin{bmatrix} 0 & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \ldots & \ldots & 0 \end{bmatrix} \tag{27}$$

and $\boldsymbol{e}^T_k$ equals the k-dimensional row vector $[0\,0\, \ldots \, 1]$ with $\boldsymbol{e}^T_1 = 1$, respectively.

Next, the system equation (23) is written as

$$\dot{\boldsymbol{x}}_n = \mathbf{A}_n \boldsymbol{x}_n + \boldsymbol{e}_n f \tag{28}$$

where

$$\boldsymbol{x}^T_n = [\, x_1 \, , \ldots , x_n \,] \tag{29}$$

If the transformation

$$\boldsymbol{x}_n = \mathbf{R}_n \boldsymbol{y}_n \tag{30}$$

with $\mathbf{R}_n$ as defined in (14) is applied to (28) and if $\boldsymbol{p}^T_n$ is any solution of the Riccati equation

$$\dot{\boldsymbol{p}}^T_n = -\boldsymbol{p}^T_n \mathbf{I}^+_{n-1} - \boldsymbol{p}^T_n \boldsymbol{e}_{n-1} \boldsymbol{p}^T_n + \boldsymbol{q}^T_n + q_{n,n} \boldsymbol{p}^T_n \tag{31}$$

then we find

$$\dot{\boldsymbol{y}}_n = \mathbf{A}_{n-1} \boldsymbol{y}_n + \boldsymbol{e}_n f \tag{32}$$

where

$$\mathbf{A}_{n-1}(t) = \begin{bmatrix} \mathbf{I}^+_{n-2} & \boldsymbol{e}_{n-2} & \boldsymbol{0} \\ \boldsymbol{q}^T_{n-1} & q_{n-1,n-1} & 1 \\ \boldsymbol{0}^T & 0 & \lambda_n \end{bmatrix} \tag{33}$$

with

$$\left.\begin{aligned} \boldsymbol{q}^T_{n-1} &= [\, p_{n,1} \, , \ldots , p_{n,n-2} \,] \\ q_{n-1,n-1} &= p_{n,n-1} \\ \lambda_n &= q_{n,n} - q_{n-1,n-1} \end{aligned}\right\} \tag{34}$$

After the application succesfully of $\mathbf{R}_{n-1}, \mathbf{R}_{n-2}, \ldots, \mathbf{R}_2$ to (32) respectively, the result of the step before, we yield accordingly to (10)

$$\dot{\boldsymbol{y}} = \mathbf{B}(t)\boldsymbol{y} + \boldsymbol{e}_n f \tag{35}$$

In this scalar case we see

$$\mathbf{B}(t) = \begin{bmatrix} \lambda_1(t) & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \ldots & \ldots & \lambda_n(t) \end{bmatrix} \tag{36}$$

This implies for the components $y_i (i = 1, 2, \ldots, n)$ of $\boldsymbol{y}$ the relations

$$\left.\begin{aligned} \dot{y}_i = \lambda_i y_i + y_{i+1} &\longleftrightarrow (D - \lambda_i) y_i = y_{i+1} \\ \dot{y}_n = \lambda_n y_n + f &\longleftrightarrow (D - \lambda_n) y_n = f \end{aligned}\right\} \tag{37}$$

As a consequence, we deduce from (37) the Cauchy-Floquet factorization

$$(D - \lambda_n)(D - \lambda_{n-1}) \ldots (D - \lambda_1) y_1 = f \tag{38}$$

Since

$$\boldsymbol{x} = \mathbf{R}_n \mathbf{R}_{n-1} \ldots \mathbf{R}_2 \boldsymbol{y} \tag{39}$$

it is easy to deduce with (14) and (17) that $\mathbf{R}_n \mathbf{R}_{n-1} \ldots \mathbf{R}_2$ is a lower triangular matrix with diagonal elements equal to 1, that

$$x_1 = y_1 \tag{40}$$

As a consequence y_1 can be replaced by x_1 and we obtain (38) as the Cauchy-Floquet factorization of (23).

Here, the dynamic eigenvalues $\lambda_k = \lambda_k(t)$ $(k = 1, 2, \ldots, n)$ are obtained by solving the Riccati-equation (31) for arbitrarily K as

$$\lambda_k = q_{k,k} - q_{k-1,k-1} \quad . \tag{41}$$

The dynamic eigenvalue λ_1 follows from the trace (22). As demonstrated in [16] en [17], the combination of (41) and the Riccati equation (14) yields the characteristic equation from which the complete spectrum $\{\lambda_1(t), \lambda_2(t), \ldots, \lambda_n(t)\}$ can be computed successively.

Finally, it follows from the factorization (38) that the so-called series D-eigenvalues, as introduced by Zhu *et.al.*[9,10], are identical to the dynamic eigenvalues.

5. CONCLUSIONS

An alternative method for the classical Cauchy-Floquet factorization of scalar polynomial differential operators is presented. It is based upon successive Riccati transformations that diagonalize the Frobenius companion system matrix. Moreover, it follows that the so-called series D-eigenvalues as introduced by Zhu *et.al.*[9,10] are identical to our dynamic eigenvalues.

6. REFERENCES

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