THE RICCATI EQUATION AS CHARACTERISTIC EQUATION FOR GENERAL LINEAR DYNAMIC SYSTEMS

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Abstract— Both linear time-invariant (LTI) and linear time-varying (LTV) systems are addressed. They are placed in a unified conceptual framework. The characteristic equation for each subclass is formulated as a Riccati equation. Where LTI-systems lead to algebraic Riccati equations, the LTV-case generalizes this result to differential Riccati equations.

I. INTRODUCTION

Linear dynamic systems may arise as variational equations in nonlinear dynamic system theory [1, 2]. If the variations move around a fixed equilibrium point in state-space, the well-established theory of linear time-invariant (LTI) systems is applicable. Therein, the concept of eigenvalues as the solutions of a characteristic equation plays a central role [3].

However, if the original nonlinear system solution describes a trajectory in state-space, the associated variational equation constitutes a linear time-varying (LTV) system. Then, it would be highly desirable to have a constructive theory to one's disposal that is compatible with the well- known theory of LTI-systems.

For example, such a unified theory could be instrumental for a rigorous analysis of the local stability properties and the noise performance of inherently nonlinear operating electronic circuits [1, 4].

Recently, the LTV approach was succesfully applied for predicting the local dynamic behavior of class-B amplifiers and translinear oscillators [5, 6].

In this article, we develop the unified theory for general linear systems a step further. Essentially, it is shown that instead of the standard formulation of the charateristic equation for LTI-systems, the (quadratic) Riccati equation is the key for generalizing LTI-concepts to LTV-concepts. In this respect, new mathematical proofs are presented.

Earlier contributions can be found in [7, 8, 9, 10, 11, 12]. In [11, 12] it is also argued that the results presented in [13], [14, 15] and [16, 17] are less general and less transparant. Moreover, in none of these references the Riccati equation is explicitely recognized as the gouverning characteristic equation.

In the next section, scalar LTI-systems are dealt with. First, it is proved that the associated polynomial characteristic equation of degree n corresponds with a $(n-1)$ vector algebraic Riccati equation. Next, the reversed is proved. Thus, the obtained Riccati equation and the conventional polynomial equation are proven to be mathematically equivalent. Then, in Section 3 vector LTI-systems are addressed. Using a suitable similarity transformation, again a $(n-1)$ vector algebraic Riccati equation is proven to be equivalent with the classical characteristic equation of a n-th order system.

Finally, in Section 4 the characteristic equation pertaining to vector LTV-systems is briefly reviewed (*cf.* [11, 12] for more details). By replacing the classical eigenvalues and the similarity transformation of Section 3 by the dynamic eigenvalues as introduced in [7, 8] and the Riccati transformation as described in [18] respectively, now vector differential Riccati equations are obtained. Comparing this result with the preceding sections, it is nicely illustated that the Riccati equation is indeed the gouverning characteristic equation for general linear dynamic systems.

II. SCALAR LTI-SYSTEMS

Consider a scalar dynamic LTI-system, described by the n-th order linear differential equation with normalized constant coefficients

$$D^n x + a_n D^{n-1} x + \cdots + a_2 D x + a_1 x = 0 \quad , \qquad (1)$$

where $x = x(t)$ denotes the output variable and $D = d/dt$. Taking modal solutions of the form $u\exp(\lambda t)$ with u an amplitude and λ an eigenfrequency as elementary solutions of (1), it follows that λ has to satisfy the polynomial equation

$$\lambda^n + a_n \lambda^{n-1} + \cdots + a_2 \lambda + a_1 = 0 \quad . \qquad (2)$$

This equation constitutes the well-known characteristic equation pertaining to scalar LTI-systems for the

unknown eigenvalues λ.
Now, the constants $r_i\,(i=0,1,\ldots,n-1)$ are introduced as

$$r_i = -\sum_{j=i+1}^{n} a_j\lambda^{j-i-1} - \lambda^{n-i} \quad , \tag{3}$$

in which λ satisfies the characteristic equation (2). Then it follows that $r_0 = 0$, while

$$\lambda r_i = a_i + r_{i-1} \quad (i=1,2,\ldots,n-1) \tag{4}$$

and

$$r_{n-1} = -a_n - \lambda \quad . \tag{5}$$

Elimination of λ from (4) and (5) yields for $i = 1,2,\ldots,n-1$ the relation

$$-r_{i-1} - r_{n-1}r_i - a_i - a_n r_i = 0 \quad , \tag{6}$$

or in vector formulation

$$-\boldsymbol{r}^T\mathbf{I}^+ - \boldsymbol{r}^T\boldsymbol{e}\boldsymbol{r}^T - \boldsymbol{a}^T - a_n\boldsymbol{r}^T = \boldsymbol{0}^T \quad , \tag{7}$$

where $\boldsymbol{r}^T = [\,r_1\,r_2\,\ldots r_{n-1}\,]$, $\boldsymbol{e}^T = [\,0\,0\ldots 1\,]$ and $\boldsymbol{a}^T = [\,a_1\,a_2\,\ldots a_{n-1}\,]$ denote $(n-1)$-dimensional row vectors (T stands for the transpose) respectively, while $\mathbf{I}^+$ is a square matrix of size $(n-1)$, given by

$$\mathbf{I}^+ = \begin{bmatrix} 0 & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \ldots & \ldots & 0 \end{bmatrix} \quad . \tag{8}$$

Equation (7) is recognized as an $(n-1)$ algebraic vector Riccati equation [19]. Note that any particular λ that satisfies (2) corresponds in view of (4) and (5) with a vector $\boldsymbol{r}$ that satisfies (7).
Next, the reverse is proved. To that aim, the state-vector $\boldsymbol{x} = [\,x_1\,x_2\,\ldots x_n\,]^T$ is introduced with components

$$x_1 = x,\; x_2 = \dot{x}_1,\; x_3 = \dot{x}_2, \ldots,\; x_n = \dot{x}_{n-1} \quad . \tag{9}$$

Then, the scalar differential equation (1) can be rewritten in the state-space description

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \mathbf{I}^+ & \boldsymbol{e} \\ -\boldsymbol{a}^T & -a_n \end{bmatrix} \boldsymbol{x} \quad . \tag{10}$$

By applying the similarity transformation

$$\boldsymbol{x} = \begin{bmatrix} \mathbf{I} & \boldsymbol{0} \\ \boldsymbol{p}^T & 1 \end{bmatrix} \boldsymbol{y} \quad , \tag{11}$$

in which $\mathbf{I}$ is the $(n-1)$ identity matrix, while $\boldsymbol{p}^T$ denotes a constant $(n-1)$ row vector with components p_i, we obtain from (10) the block-triangular system equation

$$\dot{\boldsymbol{y}} = \begin{bmatrix} \mathbf{I}^+ + \boldsymbol{e}p^T & \boldsymbol{e} \\ \boldsymbol{0}^T & \mu \end{bmatrix} \boldsymbol{y} \quad , \tag{12}$$

provided that $\boldsymbol{p}^T$ is a solution of the $(n-1)$ vector algebraic Riccati equation

$$-\boldsymbol{p}^T\mathbf{I}^+ - \boldsymbol{p}^T\boldsymbol{e}\boldsymbol{p}^T - \boldsymbol{a}^T - a_n\boldsymbol{p}^T = \boldsymbol{0}^T \quad , \tag{13}$$

while

$$\mu = -p_{n-1} - a_n \quad . \tag{14}$$

Since μ is a diagonal element in (12), it obviously represents an eigenvalue. And because the system matrices in (10) and (12) are similar, μ is a characteristic value associated with the original system (10).
However, the algebraic Riccati equation (13) not only generates the particular eigenvalue μ, but the complete eigenspectrum as well. To see this, first (14) is substituted in (13). This yields

$$\mu p_i = a_i + p_{i-1} \quad (i=1,2,\ldots,n-1) \tag{15}$$

with $p_0 = 0$. Now, it can be deduced recursively from (15) and (14) that μ is any solution of the characteristic equation (2).

III. GENERAL LTI-SYSTEMS

Next, consider the vector LTI state-equation

$$\dot{\boldsymbol{x}} = \mathbf{A}\boldsymbol{x} \quad , \tag{16}$$

in which the state-vector $\boldsymbol{x}$ is a column-vector of dimension n, while $\mathbf{A} = [a_{ij}]$ denotes a constant square system matrix of size n. Then, with the modal solutions of the preceding section, λ has to satisfy the classical characteristic equation

$$\det[\mathbf{A} - \lambda\mathbf{I}] = 0 \quad . \tag{17}$$

in which $\mathbf{I}$ denotes the identity matrix of size n. First, we show that with any particular eigenvalue λ that satisfies (17) there corresponds a solution of a $(n-1)$ vector algebraic Riccati equation.
By expanding $\det[\mathbf{A} - \lambda\mathbf{I}]$ in cofactors of the last row, we obtain

$$\det\mathbf{B} = b_{n1}B_{n1} + b_{n2}B_{n2} + \cdots + b_{nn}B_{nn} \quad , \tag{18}$$

in which $\mathbf{B} = \mathbf{A} - \lambda\mathbf{I} = [b_{ij}]$, while B_{ni} denotes a cofactor, given by

$$B_{ni} = (-1)^{n+i}\det\mathbf{M}_{ni} \quad . \tag{19}$$

Here, the minor $\mathbf{M}_{ni}$ is formed by deleting row n and column i of $\mathbf{B}$. Now, let the column vector $\boldsymbol{v}$ with $\boldsymbol{v}^T = [v_1\,v_2\,\ldots\,v_{n-1}]$ be the solution of

$$\mathbf{M}_{nn}\boldsymbol{v} = -\boldsymbol{b}_{1n} \quad \text{thus} \quad \boldsymbol{v} = -\mathbf{M}_{nn}^{-1}\boldsymbol{b}_{1n} \quad , \tag{20}$$

where $\boldsymbol{b}_{1n}^T = [b_{1n}\, b_{2n}\, \ldots\, b_{n-1,n}]$, then with Cramer's rule we obtain by careful computation for the i-th component v_i of $\boldsymbol{v}$

$$v_i = (-1)^{n+i} \det \mathbf{M}_{ni} [\det \mathbf{M}_{nn}]^{-1} \quad . \tag{21}$$

Next, on substitution of (19), (20) and (21) in (18) there results the formula

$$\det \mathbf{B} = -[\boldsymbol{b}_{n1}^T \mathbf{M}_{nn}^{-1} \boldsymbol{b}_{1n} - b_{nn}] \det \mathbf{M}_{nn} \quad , \tag{22}$$

in which $\boldsymbol{b}_{n1}^T = [b_{n1}\, b_{n2}\, \ldots\, b_{n,n-1}]$.
As a next step, the system matrix $\mathbf{A}$ is partitioned as

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \boldsymbol{a}_{12} \\ \boldsymbol{a}_{21}^T & a_{nn} \end{bmatrix} \quad , \tag{23}$$

where $\mathbf{A}_{11}$ is the $(n-1)$ left upper square block of $\mathbf{A}$, while $\boldsymbol{a}_{12}$ and $\boldsymbol{a}_{21}$ are $(n-1)$ column vectors. Then, by using (22) it follows from (17)

$$\boldsymbol{a}_{21}^T [\mathbf{A}_{11} - \lambda \mathbf{I}]^{-1} \boldsymbol{a}_{12} - (a_{nn} - \lambda) = 0 \quad . \tag{24}$$

By defining the $(n-1)$ column vector $\boldsymbol{r}$ as

$$\boldsymbol{r}^T = \boldsymbol{a}_{21}^T [\mathbf{A}_{11} - \lambda \mathbf{I}]^{-1} \quad , \tag{25}$$

it then follows from (24)

$$\lambda = a_{nn} - \boldsymbol{r}^T \boldsymbol{a}_{12} \quad . \tag{26}$$

Finally, elimination of λ from (26) and (25) yields

$$-\boldsymbol{r}^T \mathbf{A}_{11} - \boldsymbol{r}^T \boldsymbol{a}_{12} \boldsymbol{r}^T + \boldsymbol{a}_{21}^T + a_{nn} \boldsymbol{r}^T = \boldsymbol{0}^T \quad , \tag{27}$$

which is the requested $(n-1)$ vector algebraic Riccati equation for $\boldsymbol{r}$.
Analogue with the preceding section, we next prove that any particular solution $\boldsymbol{r}$ of the Riccati equation (27) again corresponds with the complete eigenspectrum. With the transformation (11) and the partition (23) of $\mathbf{A}$, we obtain from (16)

$$\dot{\boldsymbol{y}} = \begin{bmatrix} \mathbf{A}_{11} + \boldsymbol{a}_{12} \boldsymbol{p}^T & \boldsymbol{a}_{12} \\ \boldsymbol{0}^T & \mu \end{bmatrix} \boldsymbol{y} \quad , \tag{28}$$

iff $\boldsymbol{p}^T$ satisfies (27) with $\boldsymbol{r}^T$ replaced by $\boldsymbol{p}^T$, while under the same condition, μ equals the right hand side of (26). Moreover, it is readily observed that μ is an eigenvalue of the system matrix $\mathbf{A}$ in (16). Then, elimination of a_{nn} from (26) and (27) yields

$$\boldsymbol{p}^T (\mu \mathbf{I} - \mathbf{A}_{11}) + \boldsymbol{a}_{21}^T = \boldsymbol{0}^T \quad . \tag{29}$$

By combining (26) with $\boldsymbol{r}^T = \boldsymbol{p}^T$ and (29), we obtain

$$\begin{bmatrix} \boldsymbol{p}^T & -1 \end{bmatrix} \begin{bmatrix} (\mu \mathbf{I} - \mathbf{A}_{11}) & -\boldsymbol{a}_{12} \\ -\boldsymbol{a}_{21}^T & (\mu - a_{nn}) \end{bmatrix} = \boldsymbol{0}^T \quad . \tag{30}$$

Since $[\boldsymbol{p}^T \; -1] \neq \boldsymbol{0}^T$, we conclude from (30) and (23) that μ is any solution of the characteristic equation (17).

IV. GENERAL LTV-SYSTEMS

In this section we briefly consider the dynamic LTV system equation (*cf.* [9, 10] and [12] for details)

$$\dot{\boldsymbol{x}} = \mathbf{A}(t) \boldsymbol{x} \quad , \tag{31}$$

in which the elements a_{ij} of the system matrix $\mathbf{A}$ are functions of the time t. Now, the modal solutions of the preceding sections are replaced by the modal expression [20, 21]

$$\boldsymbol{x}(t) = \boldsymbol{u}(t) \exp\, [\int_0^t \lambda(\tau) d\tau] \quad . \tag{32}$$

By substitution of (32) in (31), the so called *dynamic* eigenvalue problem for the unknowns $\boldsymbol{u}(t)$ and $\lambda(t)$ follows immediately. In this context, $\boldsymbol{u}(t)$ and $\lambda(t)$ are called a *dynamic* eigenvector and a *dynamic* eigenvalue, respectively (*cf.* [8] for the nomenclature). Note that for constant systems $\lambda(t)$ reduces to the classical eigenvalue λ.
In accordance with the followed procedure in Section 3, the dynamic eigenvalue problem is solved by first partitioning the system matrix $\mathbf{A}(t)$ as displayed in(23). Next, the transformation (11), with the constant vector $\boldsymbol{p}$ replaced by a time-dependent one, is applied. In [18], such a transformation with $\boldsymbol{p} = \boldsymbol{p}(t)$ is called a Riccati transformation. With it, system (31) goes into the following block-triangular system equation (compare (28))

$$\dot{\boldsymbol{y}} = \begin{bmatrix} \mathbf{A}_{11}(t) + \boldsymbol{a}_{12}(t) \boldsymbol{p}^T(t) & \boldsymbol{a}_{12}(t) \\ \boldsymbol{0}^T & \lambda_n(t) \end{bmatrix} \boldsymbol{y} \quad , \tag{33}$$

provided that $\boldsymbol{p} = \boldsymbol{p}(t)$ is *any* solution of the $(n-1)$ vector differential Riccati equation [19]

$$\dot{\boldsymbol{p}}^T = -\boldsymbol{p}^T \mathbf{A}_{11} - \boldsymbol{p}^T \boldsymbol{a}_{12} \boldsymbol{p}^T + \boldsymbol{a}_{21}^T + a_{nn} \boldsymbol{p}^T \quad , \tag{34}$$

while the dynamic eigenvalue $\lambda_n = \lambda_n(t)$ is given by [10]

$$\lambda_n = a_{nn} - \boldsymbol{p}^T \boldsymbol{a}_{12} \quad . \tag{35}$$

By comparing (34) with (27), it is observed that the differential Riccati equation (34) is clearly the generalization of the algebraic Riccati equations in the preceding sections.
However, as demonstrated in [11, 12], in order to obtain the complete dynamic eigenspectrum $\{\lambda_1(t), \lambda_2(t), \ldots, \lambda_n(t)\}$, now a full triangularization of the time-dependent system matrix must be performed. This is accomplished by a sequence of $(n-1)$ successive Riccati transformations, each accompanied by a differential Riccati equation of decreasing order. Compared with constant systems, this reflects the apparent loss of spectral commutativity [13].

V. CONCLUSIONS

The (quadratic) Riccati equation is recognized as the gouverning characteristic equation for general linear dynamic systems. In this respect, new mathematical proofs are presented.
Together with the earlier introduced dynamic eigenvalues, it is shown that the Riccati characteristic equation places LTI- and LTV-systems in a unified conceptual framework.

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