The Characteristic Equation for Time-Varying Models of Nonlinear Dynamic Systems

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Abstract — **General linear time-varying (LTV) systems are addressed. They are considered as variational models of nonlinear dynamic system solutions. It is shown that the conventional characteristic equation for constant systems has to be replaced by a generalized one with the earlier introduced dynamic eigenvalues as unknowns. The generalized characteristic equation for scalar LTV systems follows as a special case.**

1 Introduction

Linear time-varying (LTV) systems arise as variational models in nonlinear circuit theory [1]. In this article, general LTV systems are addressed. In particular, a *generalized* characteristic equation is formulated with the earlier introduced dynamic eigenvalues as unknowns. Together with the associated dynamic eigenvectors, they constitute the exponential modal solutions as first proposed by Wu [2, 3]. Here, the adjective *dynamic* is used to distinguish LTV concepts from their conventional (static) antipodes [4, 5].

Related results can be found in [6] and [7, 8], respectively. Essentially, they are based on a factorization of a scalar polynomial differential system operator. However, in transforming the scalar system structure to a general one, serious constructive problems are encountered [9, 10].

In contrast with the work cited above, we start right a way with general LTV systems that automatically imply scalar LTV systems as a special case. Therefore, the results presented here are more general and transparant.

Our approach is based on the Riccati transformation as described in [11]. Essentially, it effectuates an order reduction and a subsequent decoupling of the original LTV system.

In the next section, the LTV models under study are placed in the context of nonlinear system theory. Then, in Section 3 the modal solutions of general LTV systems are introduced. They are characterized by a varying amplitude-vector and a varying frequency, respectively. Next, it is shown that each mode satisfies a dynamic eigenvalue problem.

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In this connection, the amplitude-vector and frequency manifest themselves as unknown dynamic eigenvector and dynamic eigenvalue, respectively.

In order to solve the dynamic eigenvalue problem, in Section 4 the LTV system is gradually triangularized by successive Riccati transformations. Under the constraint that a lower order Riccati differential equation is satisfied, in each step a next dynamic eigenvalue appears on the main diagonal.

The complete set of (quadratic) Riccati equations of decreasing order is recognized in Section 5 as the *dynamic* characteristic equation. From it, the dynamic eigenvalue spectrum follows successively.

Finally, in Section 6 scalar LTV systems are dealt with. Except from a minor error, the results presented in [8] are reproduced.

2 LTV Models

Consider a (possibly strongly) nonlinear dynamic system with state-equation

$$\dot{\boldsymbol{v}} = \boldsymbol{f}(\boldsymbol{v}, \boldsymbol{s}) \quad . \tag{1}$$

Here, $\boldsymbol{v} = \boldsymbol{v}(t)$ and $\boldsymbol{s} = \boldsymbol{s}(t)$ denote the time-dependent state-vector and input-vector, respectively, while $\boldsymbol{f}$ is a nonlinear vector function. Let $\boldsymbol{v}_C$ be a known solution of (1) belonging to a particular input $\boldsymbol{s}_C$. Further, assume that a variation $\boldsymbol{e} = \boldsymbol{e}(t)$ in $\boldsymbol{s}_C$ causes a variation $\boldsymbol{x} = \boldsymbol{x}(t)$ in $\boldsymbol{v}_C$. Then, for sufficiently small variations, a local linearization around the nominal solution trajectory C in state-space is permitted. As is well-known, the following time-varying variational state-equation is obtained from (1)

$$\dot{\boldsymbol{x}} = \mathbf{A}(t)\boldsymbol{x} + \mathbf{B}(t)\boldsymbol{e} \quad , \tag{2}$$

where $\boldsymbol{A} = [\partial \boldsymbol{f} / \partial \boldsymbol{v}]_C$ and $\boldsymbol{A} = [\partial \boldsymbol{f} / \partial \boldsymbol{s}]_C$ are Jacobians with respect to $\boldsymbol{v}$ and $\boldsymbol{s}$, respectively, both evaluated along C [1]. Thus, equation (2) describes the *local* dynamic behavior of the nonlinear system.

Since the inhomogeneous system (2) is linear in the first place, its solution is easily expressed in the solution of its homogeneous part [12]. Therefore, we will concentrate upon general LTV systems with state-equation

$$\dot{\boldsymbol{x}} = \mathbf{A}(t)\boldsymbol{x} \quad . \tag{3}$$

3 The Dynamic Eigenvalue Problem

Referring to system (3), we are looking for elementary solutions of the modal form [5, 13, 14]

$$\boldsymbol{x}(t) = \boldsymbol{u}(t)\exp[\gamma(t)] \quad , \qquad (4)$$

where $\boldsymbol{u}$ is a varying amplitude-vector, while the varying phase γ defines a varying frequency λ as

$$\lambda(t) = \dot{\gamma}(t) \quad \text{with} \quad \gamma(t) = \int_0^t \lambda(\tau)d\tau \quad . \qquad (5)$$

Substitution of (4) in the state-equation (3) yields the *dynamic* eigenvalue problem [5, 14]

$$[\mathbf{A}(t) - \lambda(t)\mathbf{I}]\boldsymbol{u}(t) = \dot{\boldsymbol{u}}(t) \quad , \qquad (6)$$

in which $\mathbf{I}$ denotes the identity matrix. In this context, the modal quantities $\boldsymbol{u}$ and λ are called a *dynamic* eigenvector and a *dynamic* eigenvalue, respectively.

Now, let system (3) be subjected to the time-dependent coordinate transformation

$$\boldsymbol{x} = \mathbf{R}(t)\boldsymbol{y} \quad , \qquad (7)$$

where $\boldsymbol{y} = \boldsymbol{y}(t)$ is the new unknown. Then, system (3) goes into another LTV system, namely

$$\dot{\boldsymbol{y}} = \mathbf{B}(t)\boldsymbol{y} \quad , \qquad (8)$$

in which $\mathbf{B} = \mathbf{R}^{-1}\mathbf{A}\mathbf{R} - \mathbf{R}^{-1}\dot{\mathbf{R}}$ [15].

4 Triangularization by the Riccati Transformation

In this section, we develop an algorithm by which a n-th order system (3) is gradually triangularized.

To that aim, we adopt the following notation

$$\dot{\boldsymbol{x}}_k = \mathbf{A}_k(t)\boldsymbol{x}_k \quad \text{for} \quad k = n, n-1, \ldots, 3, 2, \qquad (9)$$

where k refers to the dimension of the state-vector $\boldsymbol{x}_k = \boldsymbol{x}_k(t)$ and the system matrix $\mathbf{A}_k$, respectively. Next, $\mathbf{A}_k$ is partitioned as

$$\mathbf{A}_k(t) = \begin{bmatrix} \mathbf{D}_{k-1}(t) & \boldsymbol{b}_{k-1}(t) \\ \boldsymbol{c}_{k-1}^T(t) & d_{k-1}(t) \end{bmatrix} \quad . \qquad (10)$$

Here, $\mathbf{D}_{k-1}$ is the $(k-1)$ left upper square block of $\mathbf{A}_k$, $\boldsymbol{b}_{k-1}$ and $\boldsymbol{c}_{k-1}$ are $(k-1)$ column vectors, respectively, while d_{k-1} denotes a scalar. We now perform in any iteration step the coordinate transformation (*cf.* (7))

$$\boldsymbol{x}_k = \mathbf{P}_k(t)\boldsymbol{y}_k \quad , \qquad (11)$$

with $\boldsymbol{y}_k = \boldsymbol{y}_k(t)$ and where $\mathbf{P}_k$ is taken as the Riccati matrix [11]

$$\mathbf{P}_k(t) = \begin{bmatrix} \mathbf{I}_{k-1} & \boldsymbol{0} \\ \boldsymbol{p}_{k-1}^T(t) & 1 \end{bmatrix} \quad , \qquad (12)$$

in which $\mathbf{I}_{k-1}$ denotes the $(k-1)$ identity matrix while $\boldsymbol{p}_{k-1}$ is a $(k-1)$ column vector with $\boldsymbol{p}_1$ a scalar, respectively. Then in analogy with (8), we arrive at the following bock triangularized system [14]

$$\dot{\boldsymbol{y}}_k = \begin{bmatrix} \mathbf{A}_{k-1}(t) & \boldsymbol{b}_{k-1}(t) \\ \boldsymbol{0}^T & \lambda_k(t) \end{bmatrix} \boldsymbol{y}_k \quad , \qquad (13)$$

provided that $\boldsymbol{p}_{k-1} = \boldsymbol{p}_{k-1}(t)$ is *any* solution of the (quadratic) Riccati vector differential equation [16]

$$\dot{\boldsymbol{p}}_{k-1}^T = -\boldsymbol{p}_{k-1}^T\mathbf{D}_{k-1} + \boldsymbol{c}_{k-1}^T + \qquad (14)$$
$$- \boldsymbol{p}_{k-1}^T\boldsymbol{b}_{k-1}\boldsymbol{p}_{k-1}^T + d_{k-1}\boldsymbol{p}_{k-1}^T \quad ,$$

while the dynamic eigenvalue $\lambda_k(t)$ is obtained as

$$\lambda_k = d_{k-1} - \boldsymbol{p}_{k-1}^T\boldsymbol{b}_{k-1} \quad . \qquad (15)$$

Next, let the first $(k-1)$ elements of $\boldsymbol{y}_k$ define the updated state-vector $\boldsymbol{x}_{k-1}$, then the updated system matrix $\mathbf{A}_{k-1}$ in (9) follows from (13) as

$$\mathbf{A}_{k-1} = \mathbf{D}_{k-1} + \boldsymbol{b}_{k-1}\boldsymbol{p}_{k-1}^T \quad . \qquad (16)$$

By introducing the n-th order matrices $\mathbf{P}_n^{(k)}$ as

$$\mathbf{P}_n^{(k)}(t) = \begin{bmatrix} \mathbf{P}_k(t) & \boldsymbol{0} \\ \boldsymbol{0} & \mathbf{I}_{n-k} \end{bmatrix} \quad , \qquad (17)$$

it is readily observed that the Riccati matrix $\mathbf{R}_n = \mathbf{R}_n(t)$, given by

$$\mathbf{R}_n = \mathbf{P}_n^{(n)}\mathbf{P}_n^{(n-1)}\ldots\mathbf{P}_n^{(2)} \quad , \qquad (18)$$

indeed transforms system(3) into a fully triangularized system (8) with diagonal elements $\{\lambda_1(t), \lambda_2(t), \ldots, \lambda_n(t)\}$.

Finally, it follows from (12), (17) and (18) that $\det \mathbf{P}_n^{(k)}(t) = 1$, hence $\det \mathbf{R}_n(t) = 1$. As a consequence, the Riccati transformation is trace preserving, thus

$$\text{trace}\,[\mathbf{A}_n(t)] = \sum_{i=1}^n \lambda_i(t) \quad . \qquad (19)$$

5 The Dynamic Characteristic Equation

Substitution of the expression for the dynamic eigenvalues (15) into the Riccati equation (14) yields

$$\dot{\boldsymbol{p}}_{k-1}^T = \boldsymbol{p}_{k-1}^T(\lambda_k\mathbf{I}_{k-1} - \mathbf{D}_{k-1}) + \boldsymbol{c}_{k-1}^T \quad . \qquad (20)$$

If this expession is augmented with (15) we obtain in view of the partitioning in (10)

$$\dot{\boldsymbol{v}}_k^T(t) = \boldsymbol{v}_k^T(t)\,[\lambda_k(t)\mathbf{I}_k - \mathbf{A}_k(t)] \quad , \qquad (21)$$

where the k-dimensional row vector $\boldsymbol{v}_k^T$ is given by $\boldsymbol{v}_k^T = \begin{bmatrix} \boldsymbol{p}_{k-1}^T & -1 \end{bmatrix}$. Next, by taking the transpose of (21), we finally obtain with (15) the following complete set of $(n-1)$ equations ($k = n, n-1, \ldots, 2$)

$$
\left.
\begin{aligned}
[\mathbf{A}_k^T(t) - \lambda_k(t)\mathbf{I}_k]\boldsymbol{v}_k(t) &= -\dot{\boldsymbol{v}}_k(t) \quad &\text{(a)} \\
\lambda_k(t) &= d_{k-1}(t) - \boldsymbol{p}_{k-1}^T(t)\boldsymbol{b}_{k-1}(t) \quad &\text{(b)}
\end{aligned}
\right\} \quad (22)
$$

For time-invariant systems, the Riccati equation (14) admits a constant solution for $\boldsymbol{p}_{k-1}$. Then, $\dot{\boldsymbol{v}}_k = \boldsymbol{0}$ while $\boldsymbol{v}_k \neq \boldsymbol{0}$. As a consequence, equation (22.a) reduces to a homogeneous linear algebraic equation. It has non-zero solutions if $\det\left[\mathbf{A}_k^T - \lambda_k\mathbf{I}_k\right] = 0$. Thus λ_k is just a zero of the familiar characteristic equation. For this reason, equation (22) constitutes the *dynamic* characteristic equation associated with system (3).It is the *generalization* of its conventional antipode for constant systems [17].

6 The Characteristic Equation for Scalar LTV Systems

We now turn to scalar LTV systems, characterized by the n-th order linear differential equation with time-dependent coefficients $a_i = a_i(t)$ ($i = 1, 2, \ldots, n$) [6, 8]

$$
D^n x_1 + a_n D^{n-1} x_1 + \cdots + a_2 D x_1 + a_1 x_1 = 0 \quad , \quad (23)
$$

where $x_1 = x_1(t)$ is the output variable and $D = d/dt$. By introducing the new variables $\{x_2, x_3, \ldots, x_n\}$ as

$$
x_2 = \dot{x}_1, \, x_3 = \dot{x}_2, \ldots, x_n = \dot{x}_{n-1} \quad , \quad (24)
$$

the scalar equation (23) is recast in the state-space description (3) with companion matrix $\mathbf{A} = \mathbf{A}_n$, given by

$$
\mathbf{A}_n(t) = \begin{bmatrix} \mathbf{I}_{n-1}^+ & \boldsymbol{e}_{n-1} \\ \boldsymbol{c}_{n-1}^T(t) & d_{n-1}(t) \end{bmatrix} \quad (25)
$$

with

$$
\boldsymbol{c}_{n-1}^T = -[a_1 \, a_2 \ldots a_{n-1}] \, , \, d_{n-1} = -a_n \, , \quad (26)
$$

while $\mathbf{I}_k^+$ denotes a square matrix of size k with zero elements except the elements on the l-th row and $(l+1)$-th column ($l = 1, 2, \ldots, k-1$), which are 1, with $\mathbf{I}_1^+ = 0$ and $\boldsymbol{e}_k^T$ equals the k-dimensional row vector $[0\,0\,\ldots\,1]$ with $\boldsymbol{e}_1^T = 1$, respectively.

We now proceed as in the preceding sections. On account of (16) $\mathbf{A}_k$ ($k = n-1, n-2, \ldots, 3, 2$) is obtained as in (25) with n replaced by k, while

$$
\boldsymbol{c}_{k-1}^T = [p_{k,1} \, p_{k,2} \ldots p_{k,k-1}] \, , \, d_{k-1} = p_{k,k} \, , \quad (27)
$$

where $p_{k,i}$ denotes the i-th component of $\boldsymbol{p}_k$. Furthermore, equation (20) becomes ($k = n, n-1, \cdots, 3, 2$)

$$
\dot{\boldsymbol{p}}_{k-1}^T = \boldsymbol{p}_{k-1}^T(\lambda_k\mathbf{I}_{k-1} - \mathbf{I}_{k-1}^+) + \boldsymbol{c}_{k-1}^T \quad , \quad (28)
$$

where $\boldsymbol{c}_{k-1}^T$ is given by (26) and (27), respectively.

Moreover, the dynamic eigenvalues are obtained form (15) as

$$
\lambda_k = p_{k,k} - p_{k-1,k-1} \quad , \quad (29)
$$

in which $p_{n,n}$ has to be replaced by $-a_n$, while λ_1 follows from the trace (19).

We will now show by example for $n = 3$ that the solution $\boldsymbol{p}_{k-1}$ of the Riccati-equation (14) can be eliminated from (28) and (29), resulting in a set of nonlinear differential equations from which the dynamic eigenvalues $\lambda_1 = \lambda_1(t)$, $\lambda_2 = \lambda_2(t)$ and $\lambda_3 = \lambda_3(t)$ can be solved successively. Starting with $k = 3$, we obtain from (29)

$$
\lambda_3 = -a_3 - p_{2,2} \quad , \quad (30)
$$

while (28) falls apart into two coupled scalar differential equations, given by

$$
\left.
\begin{aligned}
\dot{p}_{2,1} &= \lambda_3 p_{2,1} - a_1 \quad &\text{(a)} \\
\dot{p}_{2,2} &= \lambda_3 p_{2,2} - p_{2,1} - a_2 \quad &\text{(b)}
\end{aligned}
\right\} \quad (31)
$$

For $k = 2$ it follows from (29) and (28) respectively

$$
\lambda_2 = p_{2,2} - p_1 \quad , \quad (32)
$$

$$
\dot{p}_1 = \lambda_2 p_1 + p_{2,1} \quad . \quad (33)
$$

Elimination of λ_3 from (19) and (30) yields

$$
p_{2,2} = \lambda_1 + \lambda_2 \quad , \quad (34)
$$

while a subsequent substitution in (32) gives $p_1 = \lambda_1$, which on its turn leads with (33) to

$$
p_{2,1} = \dot{\lambda}_1 - \lambda_1\lambda_2 \quad . \quad (35)
$$

Next, by substitution of (34) and (35) in (31.b), while using (19) we obtain

$$
\dot{\lambda}_2 + \lambda_2^2 + \lambda_2\lambda_1 + a_3(\lambda_2 + \lambda_1) + \lambda_1^2 + 2\dot{\lambda}_1 + a_2 = 0, \quad (36)
$$

from which λ_2 can be solved, provided that λ_1 can be solved first. To that aim, (35) is substituted in (31.a), while again (19) is used to eliminate λ_3 from the result. Then, if (36) is multiplied by λ_1 and subsequently summed together with the above mentioned result, the requested equation for λ_1 becomes

$$
\ddot{\lambda}_1 + 3\lambda_1\dot{\lambda}_1 + \lambda_1^3 + a_3(\lambda_1^2 + \dot{\lambda}_1) + a_2\lambda_1 + a_1 = 0. \quad (37)
$$

Finally, for known solutions λ_1 and λ_2, λ_3 is found from the trace (19).

It is noted that the equations (19), (37) and (36) are also produced in [8]. However, the factor 2 in (36) is missing in equation (7.3) of [8].

Compared with (22), it is observed that elimination of the Riccati solution doesn't lead to increased transparancy. Therefore, instead of the set (19), (37) and (36) we will call the set of equations (22) the dynamic characteristic equation. Then it is also immediately clear that the Riccati-equation is the key for generalizing the conventional theory pertaining to constant systems to a consistent LTV system theory [18].

7 Conclusions

The local dynamic behavior of nonlinear dynamic system solutions is described by LTV equations. As general LTV systems are concerned, the dynamic characteristic equation is formulated. It is recognized as the complete augmented set of (quadratic) Riccati differential equations of decreasing order. From it, the dynamic eigenvalues can be computed successively .

As scalar LTV systems are concerned, the results presented in [8] are reproduced.

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