THE DYNAMIC CHARACTERISTIC EQUATION

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Abstract—*General linear time-varying (LTV) systems are addressed. They arise as small-signal models in nonlinear electronics. It is shown that the conventional characteristic equation for constant systems has to be replaced by a generalized one with the earlier introduced dynamic eigenvalues as unknowns. A simple example clearly illustrates the difference between the conventional eigenvalues and their dynamic counterparts.*

I. INTRODUCTION

Linear time-varying (LTV) systems arise as small-signal models in nonlinear circuit theory [1], [2]. Recently, they were successfully applied in predicting the local dynamic behavior of class-B amplifiers [3], oscillators [4] and dynamic translinear circuits [5], respectively.

In this article, general LTV systems are addressed. In particular, the *generalized* characteristic equation is formulated with the earlier introduced dynamic eigenvalues as unknowns. Together with the associated dynamic eigenvectors, they constitute the exponential modal solutions as first proposed by Wu [6], [7]. At this place it is remarked that we have adopted the adjective *dynamic* to distinguish LTV concepts from their conventional (static) antipodes [8], [9].

As scalar LTV systems are concerned, Kamen [10] obtained modal solutions by factoring the associated polynomial differential system operator. For the second order case, the factorization was explicitly given in terms of a solution of the Riccati differential equation.

A related approach was followed by Zhu *et al.* [11], [12]. These authors found modal solutions based on a scalar differential operator factorization due to Cauchy-Floquet. Inspired by the classical series and parallel canonical realizations of constant scalar systems, they introduced two types of (interrelated) time-dependent eigenvalues, each type satisfying nonlinear constraint equations. Then, the scalar system is written in vector form with an associated time-dependent Frobenius companion system matrix. However, in transforming this particular system structure to a general one, serious constructive problems were encountered [13], [14].

In contrast with the work cited above, we start right a way with general LTV systems that automatically imply scalar LTV systems as a special case. Therefore, our results are more general and transparant.

The approach presented here, is based on the Riccati transformation as described in [15]. Essentially, it effectuates an appropriate order reduction and a subsequent decoupling of the original LTV system.

In Section II the modal solutions of general LTV systems are introduced. They are characterized by a varying amplitude-vector and a varying frequency, respectively. Next, it is shown that each mode satisfies a dynamic eigenvalue problem. In this connection, the amplitude-vector and frequency manifest themselves as unknown dynamic eigenvector and dynamic eigenvalue, respectively.

In order to solve the dynamic eigenvalue problem, in Section III the LTV system is gradually triangularized by successive Riccati transformations. Under the constraint that a lower order Riccati differential equation is satisfied, in each step a next dynamic eigenvalue appears on the main diagonal. In Section IV the complete set of (quadratic) Riccati equations of decreasing order is recognized as the *dynamic* characteristic equation. From it, the *ordered* set of dynamic eigenvalues follows by successive computation.

Finally, Section V presents an elementary example that clearly illustrates the difference between the conventional and dynamic eigenvalues, respectively.

(**??**) belonging to a particular input

II. THE DYNAMIC EIGENVALUE PROBLEM

Consider the n-dimensional homogeneous LTV system

$$\dot{\boldsymbol{x}} = \mathbf{A}(t)\boldsymbol{x} \quad . \tag{1}$$

We are looking for elementary solutions of the modal form [9], [16], [17]

$$\boldsymbol{x}(t) = \boldsymbol{u}(t)\exp[\gamma(t)] \quad , \tag{2}$$

where $\boldsymbol{u}$ denotes a varying amplitude-vector, while the varying phase γ defines a varying frequency λ as

$$\lambda(t) = \dot{\gamma}(t) \quad \text{with} \quad \gamma(t) = \int_0^t \lambda(\tau)d\tau \quad . \tag{3}$$

Substitution of (2) in the state-equation (1) yields the *dynamic* eigenvalue problem [9], [17]

$$[\mathbf{A}(t) - \lambda(t)\mathbf{I}]\boldsymbol{u}(t) = \dot{\boldsymbol{u}}(t) \quad , \tag{4}$$

in which $\mathbf{I}$ denotes the identity matrix. In this context, the modal quantities $\boldsymbol{u}$ and λ are called a *dynamic* eigenvector and a *dynamic* eigenvalue, respectively. In order to solve (4) for $\boldsymbol{u}$ and λ, system (1) is subjected to the time-dependent coordinate transformation

$$\boldsymbol{x} = \mathbf{R}(t)\boldsymbol{y} \quad , \tag{5}$$

where $\boldsymbol{y} = \boldsymbol{y}(t)$ is the new unknown. Then, system (1) goes into another LTV system, namely

$$\dot{\boldsymbol{y}} = \mathbf{B}(t)\boldsymbol{y} \quad , \tag{6}$$

in which the system matrix $\mathbf{B}$ is given by [18]

$$\mathbf{B} = \mathbf{R}^{-1}\mathbf{A}\mathbf{R} - \mathbf{R}^{-1}\dot{\mathbf{R}} \quad . \tag{7}$$

It is easily shown that system (1) and (6) share the same dynamic eigenvalues. Therefore, the system matrices $\mathbf{A}$ and $\mathbf{B}$ are called *dynamically* similar [9]. In the next section, we construct a coordinate transformation matrix $\mathbf{R}$ by which system (1) is gradually triangularized. As shown earlier, in each triangularization step, a next dynamic eigenvalue appears on the main diagonal. Once the system is fully triangularized, the complete dynamic eigenvalue-spectrum is known. Then, the associated dynamic eigenvectors are easily found by straightforward integration and back substitution [17]. This motivates our exclusive attention for the relations which have to be satisfied by the dynamic eigenvalues.

III. TRIANGULARIZATION BY THE RICCATI TRANSFORMATION

In this section, we develop an algorithm by which system (1) is gradually triangularized. To that aim, we adopt the following notation

$$\dot{\boldsymbol{x}}_k = \mathbf{A}_k(t)\boldsymbol{x}_k \quad \text{for} \quad k = n, n-1, \ldots, 3, 2, \tag{8}$$

where k refers to the dimension of the state-vector $\boldsymbol{x}_k = \boldsymbol{x}_k(t)$ and the system matrix $\mathbf{A}_k$, respectively. Next, in any iteration step $\mathbf{A}_k$ is partitioned as

$$\mathbf{A}_k(t) = \begin{bmatrix} \mathbf{D}_{k-1}(t) & \boldsymbol{b}_{k-1}(t) \\ \boldsymbol{c}_{k-1}^T(t) & d_{k-1}(t) \end{bmatrix} \quad . \tag{9}$$

Here, $\mathbf{D}_{k-1}$ is the $(k-1)$ left upper square block of $\mathbf{A}_k$, $\boldsymbol{b}_{k-1}$ and $\boldsymbol{c}_{k-1}$ are $(k-1)$ column vectors, respectively, while d_{k-1} denotes a scalar (T stands for the transpose). We now perform in any iteration step the coordinate transformation (*cf.* (5))

$$\boldsymbol{x}_k = \mathbf{P}_k(t)\boldsymbol{y}_k \quad , \tag{10}$$

with $\boldsymbol{y}_k = \boldsymbol{y}_k(t)$ and where $\mathbf{P}_k$ is taken as the Riccati matrix [15]

$$\mathbf{P}_k(t) = \begin{bmatrix} \mathbf{I}_{k-1} & \boldsymbol{0} \\ \boldsymbol{p}_{k-1}^T(t) & 1 \end{bmatrix} \quad , \tag{11}$$

in which $\mathbf{I}_{k-1}$ denotes the $(k-1)$ identity matrix while $\boldsymbol{p}_{k-1}$ is a $(k-1)$ column vector with $\boldsymbol{p}_1$ a scalar, respectively. Then in analogy of (6), we arrive on account of (7) at the following block triangularized LTV system

$$\dot{\boldsymbol{y}}_k = \begin{bmatrix} \mathbf{A}_{k-1}(t) & \boldsymbol{b}_{k-1}(t) \\ \boldsymbol{0}^T & \lambda_k(t) \end{bmatrix} \boldsymbol{y}_k \quad , \tag{12}$$

where we used

$$\mathbf{P}_k^{-1}(t) = \begin{bmatrix} \mathbf{I}_{k-1} & \boldsymbol{0} \\ -\boldsymbol{p}_{k-1}^T(t) & 1 \end{bmatrix} \quad , \tag{13}$$

and provided that $\boldsymbol{p}_{k-1} = \boldsymbol{p}_{k-1}(t)$ is *any* solution of the (quadratic) Riccati vector differential equation [19]

$$\dot{\boldsymbol{p}}_{k-1}^T = \\ -\boldsymbol{p}_{k-1}^T\mathbf{D}_{k-1} + \boldsymbol{c}_{k-1}^T - \boldsymbol{p}_{k-1}^T\boldsymbol{b}_{k-1}\boldsymbol{p}_{k-1}^T + d_{k-1}\boldsymbol{p}_{k-1}^T , \tag{14}$$

while the dynamic eigenvalue $\lambda_k = \lambda_k(t)$ is obtained as

$$\lambda_k = d_{k-1} - \boldsymbol{p}_{k-1}^T\boldsymbol{b}_{k-1} \quad . \tag{15}$$

Next, let the first $(k-1)$ elements of $\boldsymbol{y}_k$ define the updated state-vector $\boldsymbol{x}_{k-1}$, then the updated system matrix $\mathbf{A}_{k-1}$ in (8) follows from the result obtained in (12) as

$$\mathbf{A}_{k-1} = \mathbf{D}_{k-1} + \boldsymbol{b}_{k-1}\boldsymbol{p}_{k-1}^T \quad . \tag{16}$$

At the end of the iteration process, we finally arrive at

$$\dot{\boldsymbol{y}}_2 = \begin{bmatrix} \lambda_1(t) & \boldsymbol{b}_1(t) \\ 0 & \lambda_2(t) \end{bmatrix} \boldsymbol{y}_2 \quad , \tag{17}$$

where $\boldsymbol{b}_1$ defines a scalar.
By introducing the n-th order matrices $\mathbf{P}_n^{(k)}$ as

$$\mathbf{P}_n^{(k)}(t) = \begin{bmatrix} \mathbf{P}_k(t) & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-k} \end{bmatrix} \quad , \qquad (18)$$

it is readily observed that the Riccati matrix $\mathbf{R}_n = \mathbf{R}_n(t)$, given by

$$\mathbf{R}_n = \mathbf{P}_n^{(n)} \mathbf{P}_n^{(n-1)} \ldots \mathbf{P}_n^{(2)} \quad , \qquad (19)$$

indeed transforms system(1) into a fully triangularized system (6) with diagonal elements $\{\lambda_1(t), \lambda_2(t), \ldots, \lambda_n(t)\}$.
Finally, it follows from (11), (18) and (19) that

$$\det \mathbf{P}_n^{(k)}(t) = 1 \quad \text{hence} \quad \det \mathbf{R}_n(t) = 1 \quad . \qquad (20)$$

As a consequence, the Riccati transformation is trace preserving, thus

$$\text{trace}\,[\mathbf{A}_n(t)] = \sum_{i=1}^{n} \lambda_i(t) \quad . \qquad (21)$$

IV. THE DYNAMIC CHARACTERISTIC EQUATION

Substitution of the expression for the dynamic eigenvalues (15) into the Riccati equation (14) yields

$$\dot{\boldsymbol{p}}_{k-1}^T = \boldsymbol{p}_{k-1}^T (\lambda_k \mathbf{I}_{k-1} - \mathbf{D}_{k-1}) + \boldsymbol{c}_{k-1}^T \quad . \qquad (22)$$

If this expession is augmented with (15) we obtain

$$\dot{\boldsymbol{v}}_k^T = \boldsymbol{v}_k^T \begin{bmatrix} [\lambda_k \mathbf{I}_{k-1} - \mathbf{D}_{k-1}] & -\boldsymbol{b}_{k-1} \\ -\boldsymbol{c}_{k-1}^T & (\lambda_k - d_{k-1}) \end{bmatrix} \quad , \qquad (23)$$

where the k-dimensional row vector $\boldsymbol{v}_k^T$ is given by

$$\boldsymbol{v}_k^T = \begin{bmatrix} \boldsymbol{p}_{k-1}^T & -1 \end{bmatrix} \quad . \qquad (24)$$

In view of the displayed partitioning of $\mathbf{A}_k$ in (9), equation (23) can be put together as

$$\dot{\boldsymbol{v}}_k^T(t) = \boldsymbol{v}_k^T(t)\,[\lambda_k(t)\mathbf{I}_k - \mathbf{A}_k(t)] \quad . \qquad (25)$$

Next, by taking the transpose of (25), we finally obtain with (15) the following complete set of $(n-1)$ equations $(k = n, n-1, \ldots, 2)$

$$\left. \begin{aligned} &[\mathbf{A}_k^T(t) - \lambda_k(t)\mathbf{I}_k]\boldsymbol{v}_k(t) = -\dot{\boldsymbol{v}}_k(t) \quad &\text{(a)} \\ &\lambda_k(t) = d_{k-1}(t) - \boldsymbol{p}_{k-1}^T(t)\boldsymbol{b}_{k-1}(t) \quad &\text{(b)} \end{aligned} \right\} \qquad (26)$$

For time-invariant systems, the Riccati equation (14) admits a constant solution for $\boldsymbol{p}_{k-1}$. Then, $\dot{\boldsymbol{v}}_k = \boldsymbol{0}$ while we always have $\boldsymbol{v}_k \neq \boldsymbol{0}$. As a consequence, equation (26.a) reduces to a homogeneous linear algebraic equation. It has non-zero solutions if

$$\det\left[\mathbf{A}_k^T - \lambda_k \mathbf{I}_k\right] = 0 \text{ for } k = n, n-1, \ldots, 2 \,. \qquad (27)$$

Since the system matrices in (8)and (12)are similar for constant systems, they have identical (conventional) eigenvalues. Hence, by repeated use of $\det[\mathbf{A}_k - \lambda \mathbf{I}_k] = \det[\mathbf{A}_{k-1} - \lambda \mathbf{I}_{k-1}](\lambda_k - \lambda) = 0$, it follows that the familar characteristic equation for system (1)with constant system matrix, namely $\det[\mathbf{A}_n - \lambda \mathbf{I}_n] = 0$, is satisfied by the eigenvalues $\lambda = \lambda_k$ as they are defined in (26).For this reason, equation (26)constitutes the *dynamic* characteristic equation associated with system (1).It is the *generalization* of its conventional antipode for constant systems.
In summary, starting with $k = n$ the dynamic eigenvalues $\lambda_k(t)$ are for decreasing k computed as follows: 1. substitute (22.b) in (22.a) and find a particular solution $\boldsymbol{p}_{k-1}(t)$ for the resulting Riccati equation, 2. find $\lambda_k(t)$ by back substitution of $\boldsymbol{p}_{k-1}(t)$ in (22.b) and finally 3. find $\lambda_1(t)$ from the trace (21). The result is the *ordered* set of dynamic eigenvalues $\{\lambda_1(t), \lambda_2(t), \cdots, \lambda_n(t)\}$.

V. CONVENTIONAL VERSUS DYNAMIC EIGENVALUES

As an example, consider the two-dimensional LTV system

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -\omega_0 \tan(\omega_0 t) & 1 \\ 0 & \omega_0 \tan(\omega_0 t) \end{bmatrix} \boldsymbol{x} \quad , \qquad (28)$$

in which ω_0 is a positive constant. By direct inspection, the time-varying but conventional eigenvalues of the system matrix are

$$\lambda_1(t) = -\omega_0 \tan(\omega_0 t)\,, \lambda_2(t) = \omega_0 \tan(\omega_0 t)\,. \qquad (29)$$

Thus, for almost any time we always have one real positive eigenvalue. In the conventional perspective this would predict unstable system behavior. Yet, it is easily verified that

$$\boldsymbol{x} = \begin{bmatrix} \omega_0^{-1} \sin(\omega_0 t) & (\cos(\omega_0 t))^{-1} \end{bmatrix}^T \qquad (30)$$

is a stable solution of (28). In fact, system (28) turns out to be a triangularized LTV state-space description of the (neutrally stable) harmonic oscillator [17]

$$\dot{\boldsymbol{y}} = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix} \boldsymbol{y} \quad , \qquad (31)$$

with conventional eigenvalues $\lambda_{1,2} = \pm j\omega_0$.
The example underlines the well-known fact that the conventional eigenvalues give no indication of the stability properties of an associated LTV system. Except for a triangular or diagonal LTV system, the eigenvalues of the system *matrix* and the eigenvalues of the *system* itself (the dynamic eigenvalues) do not coincide, as they always do for constant systems.

VI. CONCLUSIONS

The local dynamic behavior of nonlinear dynamic system solutions is described by LTV equations. As general LTV systems are concerned, the dynamic characteristic equation is formulated. It is recognized as the complete augmented set of (quadratic) Riccati differential equations of decreasing order, necessary for triangularization the LTV system. From it, the dynamic eigenvalues can be successively computed. As in the classical context where an algebraic characteristic equation has to be solved, solutions of its dynamical counterpart are not *per se* easy to obtain. However, its theoretical significance is obvious.

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