A COMPLEMENTARY VIEW ON TIME-VARYING SYSTEMS

F.L. Neerhoff and P. van der Kloet

Delft University of Technology
Faculty of Information Technology and Systems
Electronics Research Laboratory
Mekelweg 4, 2628 CD Delft, The Netherlands
email: F.L.Neerhoff@its.tudelft.nl

ABSTRACT

This contribution is complementary to [1] in that it takes a time-varying mode-vector solution as an *a priori* assumption. The associated dynamic eigenvalue problem is solved by triangularizing the system equations, again accomplished by successive Riccati transforms. It is explicitly shown that equal eigenvalues do not give rise to the Jordan form. Although the mode-vectors are not uniquely determined, it is demonstrated by example that different representations yield identical transition matrices, as it should.

1. INTRODUCTION

Where in [1] time-varying modes resulted from a direct factorization of the linear time-varying (LTV) system equations, we now take the complementary point of view. That is, a time-varying mode is right from the beginning assumed to be an elementary solution of a general LTV system. This is in agreement with the classical approach for constant systems [2, 3]. Then, it is demonstrated that the classical solution procedure is naturally generalized to the LTV context.

Related results can be found in [4, 5, 6, 7, 8]. However, as pointed out in the accompanying paper [1], our results are more general and transparant.

In the next section, the dynamic eigenvalue problem is formulated and the conventional concept of similarity is generalized. Next, in Section 3, the system equations are triangularized by successive Riccati transforms [9]. The dynamic eigenvalue problem is subsequently solved in Section 4. It is shown that no exception has to be made for equal (dynamic) eigenvalues. Finally, in Section 5 some aspects of the theory are illustrated for second order LTV systems.

2. THE DYNAMIC EIGENVALUE PROBLEM

Consider the n-dimensional homogeneous LTV system

$$\dot{\boldsymbol{x}} = \mathbf{A}(t)\boldsymbol{x} \quad , \tag{1}$$

where $\mathbf{A}$ denotes a known system matrix with elements that are continuous functions of the time t and $\boldsymbol{x} = \boldsymbol{x}(t)$ an unknown vector variable, respectively. As is well-known, the transition matrix of system (1) is *uniquely* determined [10].

We are looking for elementary solutions of the modal form [1]

$$\boldsymbol{x}(t) = \boldsymbol{u}(t)\exp[\gamma(t)] \quad , \tag{2}$$

where $\boldsymbol{u}$ is interpreted as a varying amplitude-vector, while the varying phase γ is related to a varying frequency λ, given by

$$\lambda(t) = \dot{\gamma}(t) \quad \text{with} \quad \gamma(t) = \int_0^t \lambda(\tau)d\tau \quad . \tag{3}$$

Substitution of (2) in (1) yields the dynamic eigenvalue problem (*cf.* [11])

$$[\mathbf{A}(t) - \lambda(t)\mathbf{I}]\boldsymbol{u}(t) = \dot{\boldsymbol{u}}(t) \quad , \tag{4}$$

where $\mathbf{I}$ is the n-dimensional identity matrix. In the context of (4), the physical quantities $\boldsymbol{u}$ and λ are called a dynamic eigenvector and a dynamic eigenvalue, respectively. In order to solve (4), we subject system (1) to the time-varying algebraic coordinate transform

$$\boldsymbol{x} = \mathbf{L}(t)\boldsymbol{y} \quad , \tag{5}$$

where $\boldsymbol{y} = \boldsymbol{y}(t)$ is the new unknown. Then, system (1) goes into another LTV-system, viz.

$$\dot{\boldsymbol{y}} = \mathbf{B}(t)\boldsymbol{y} \quad , \tag{6}$$

in which the system matrix $\mathbf{B}$ is obtained as [12]

$$\mathbf{B} = \mathbf{L}^{-1}\mathbf{A}\mathbf{L} - \mathbf{L}^{-1}\dot{\mathbf{L}} \quad . \tag{7}$$

In turn, the transformed dynamic eigenvalue problem reads

$$[\mathbf{B}(t) - \lambda(t)\mathbf{I}]\boldsymbol{v}(t) = \dot{\boldsymbol{v}}(t) \quad , \tag{8}$$

where

$$\boldsymbol{v} = \mathbf{L}^{-1}\boldsymbol{u} \quad , \tag{9}$$

while $\boldsymbol{y}$ has the modal form

$$\boldsymbol{y}(t) = \boldsymbol{v}(t)\exp[\gamma(t)] \quad . \tag{10}$$

From (4) and (8) it is observed that the dynamic eigenvalues $\lambda(t)$ are *preserved* under the coordinate transform $\mathbf{L}(t)$. For this reason, the system matrices $\mathbf{A}$ and $\mathbf{B}$ are called *dynamically* similar [13]. In the particular case that $\mathbf{L}$ is constant, the matrices become similar in the conventional (static) sense.

3. TRIANGULARIZATION BY SUCCESSIVE RICCATI TRANSFORMS

Turning back to the dynamic eigenvalue problem (4), we first construct a coordinate transform that triangularize system (1). To that aim, we take $\mathbf{L}$ in (5) as the Riccati transform [9]

$$\mathbf{P}_1(t) = \begin{bmatrix} \mathbf{I} & \boldsymbol{0} \\ \boldsymbol{p}_1^T(t) & 1 \end{bmatrix} \quad , \tag{11}$$

where $\mathbf{I}$ is the $(n-1)$ identity matrix and $\boldsymbol{p}_1$ an $(n-1)$ column vector, respectively, while T stands for the transpose. Next, the system matrix $\mathbf{A}$ in (1) is partitioned as

$$\mathbf{A}(t) = \begin{bmatrix} \mathbf{A}_{11}(t) & \boldsymbol{a}_{12}(t) \\ \boldsymbol{a}_{21}^T(t) & a_{nn}(t) \end{bmatrix} \quad . \tag{12}$$

Here, $\mathbf{A}_{11}$ is the $(n-1)$ upper left square block of $\mathbf{A}$ while $\boldsymbol{a}_{12}$ and $\boldsymbol{a}_{21}$ are $(n-1)$ column vectors, respectively. Now, if $\boldsymbol{p}_1$ satisfies the (quadratic) Riccati vector differential equation (*cf.* [1])

$$\dot{\boldsymbol{p}}_1^T = -\boldsymbol{p}_1^T \mathbf{A}_{11} + \boldsymbol{a}_{21}^T - \boldsymbol{p}_1^T \boldsymbol{a}_{12} \boldsymbol{p}_1^T + a_{nn} \boldsymbol{p}_1^T \quad , \tag{13}$$

system (1) goes with (7) into the following block triangular form

$$\dot{\boldsymbol{y}} = \begin{bmatrix} [\mathbf{A}_{11}(t) + \boldsymbol{a}_{12}(t)\boldsymbol{p}_1^T(t)] & \boldsymbol{a}_{12}(t) \\ \boldsymbol{0}^T & c_{nn}(t) \end{bmatrix} \boldsymbol{y} \quad , \tag{14}$$

where the scalar function $c_{nn} = c_{nn}(t)$ is given by

$$c_{nn} = a_{nn} - \boldsymbol{p}_1^T \boldsymbol{a}_{12} \quad . \tag{15}$$

It is clear that the procedure can be repeated with respect to the $(n-1)$ square matrix $[\mathbf{A}_{11} + \boldsymbol{a}_{12}\boldsymbol{p}_1^T]$. This is accomplished by a next Riccati transform, given by

$$\mathbf{P}_2(t) = \begin{bmatrix} \mathbf{I} & \boldsymbol{0} & \vdots & \boldsymbol{0} \\ \boldsymbol{p}_2^T(t) & 1 & \vdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \boldsymbol{0}^T & 0 & \vdots & 1 \end{bmatrix} \quad , \tag{16}$$

in which $\boldsymbol{p}_2$ is an $(n-2)$ column vector and $\mathbf{I}$ the $(n-2)$ identity matrix, respectively. After $(n-1)$ succesive Riccati transforms we finally arrive at

$$\boldsymbol{x} = \mathbf{R}(t)\boldsymbol{z} \quad \text{with} \quad \mathbf{R} = \mathbf{P}_1\mathbf{P}_2\cdots\mathbf{P}_{n-1} \quad , \tag{17}$$

by which system (1) is fully triangularized in the variable $\boldsymbol{z} = \boldsymbol{z}(t)$. In (17) the last Riccati transform is given by

$$\mathbf{P}_{n-1}(t) = \begin{bmatrix} 1 & 0 & \vdots & \\ & & \vdots & \mathbf{0} \\ p_{n-1}(t) & 1 & \vdots & \\ \cdots & \cdots & \cdots & \cdots \\ & \mathbf{0} & \vdots & \mathbf{I} \end{bmatrix} \quad , \tag{18}$$

where p_{n-1} is a scalar and $\mathbf{I}$ the $(n-2)$ identity matrix, respectively. Note that $\det \mathbf{P}_i = 1$, hence $\det \mathbf{R} = 1$.
If $c_{ij} = c_{ij}(t)$ are the elements of the governing system matrix $\mathbf{C}$, we thus have

$$\dot{\boldsymbol{z}} = \mathbf{C}(t)\boldsymbol{z}\,;\, c_{ij} = 0 \text{ for } i > j \; (i,j = 1,2,\ldots,n)\,. \tag{19}$$

4. SOLVING THE DYNAMIC EIGENVALUE PROBLEM

Since the system matrices $\mathbf{A}$ and $\mathbf{C}$ are dynamically similar, the solution of (19) can be written as a linear combination of the mode-vectors

$$\boldsymbol{z}_j(t) = \boldsymbol{w}_j(t)\exp[\gamma_j(t)] \quad (j = 1,2,\ldots,n) \quad , \tag{20}$$

in which the dynamic eigenvalues $\lambda_j = \dot{\gamma}_j$ are precisely the ones pertaining to system (1). In analogy with (8), the transformed dynamic eigenvalue problem reads

$$[\mathbf{C}(t) - \lambda_j(t)\mathbf{I}]\boldsymbol{w}_j(t) = \dot{\boldsymbol{w}}_j \quad (j = 1,2,\ldots,n) \quad , \tag{21}$$

where the dynamic eigenvectors $\boldsymbol{w}_j$ belong to λ_j. In view of the upper triangular structure of $\mathbf{C}$, it is readily observed that

$$\lambda_j(t) = c_{jj}(t) \quad (j = 1,2,\ldots,n) \tag{22}$$

satisfies (21), while $\boldsymbol{w}_j$ $(j = 1,2,\ldots,n)$ can be chosen as the linearly independent set

$$\boldsymbol{w}_j = [\, w_{1j}\; w_{2j}\; \ldots w_{j-1,j}\; 1\, 0\, 0 \ldots 0\,]^T \quad . \tag{23}$$

Next, the non-zero elements $w_{ij} = w_{ij}(t)$ of $\boldsymbol{w}_j$ follow from back substitution of (22) and (23) in (21) as the solutions of the linear differential equation with time dependent coefficients

$$\dot{w}_{ij} = (\lambda_i - \lambda_j)w_{ij} + \sum_{k=i+1}^{j} c_{ik}w_{kj} \;\; (i \le j)\,. \tag{24}$$

Straightforward integration of (24), whereby the method of variation of parameters is used, yields the closed expression

$$w_{ij}(t) = \int_0^t k_{ij}(t,\tau)\sum_{k=i+1}^{j} c_{ik}(\tau)w_{kj}(\tau)d\tau \; (i \le j)\,, \tag{25}$$

with

$$k_{ij}(t,\tau) = \exp[\gamma_i(t,\tau) - \gamma_j(t,\tau)] \quad , \tag{26}$$

where we wrote

$$\gamma_i(t,\tau) = \gamma_i(t) - \gamma_i(\tau) \quad . \tag{27}$$

From (25), the unknown elements w_{ij} of the dynamic eigenvectors $\boldsymbol{w}_j$ $(j = 1,2,\ldots,n)$ follow explicitly by successively taking $i = j-1,\ldots,1$. (Note that $w_{jj} = 1$.)
It is clearly observed from (25) and (26) that no exception needs to be made for *equal* dynamic eigenvalues. For example, if

$$\lambda_i(t) = \lambda_{i+1}(t) = \lambda_{i+2}(t) \quad , \tag{28}$$

we obtain from (25)

$$w_{i,i+1}(t) = \int_0^t c_{i,i+1}(\tau)d\tau \quad , \tag{29}$$

and next

$$w_{i,i+2}(t) = \int_0^t \{c_{i,i+1}(\tau)\int_0^\tau \{c_{i+1,i+2}(\sigma)d\sigma\}d\tau + \int_0^t c_{i,i+2}(\tau)d\tau\,. \tag{30}$$

Even in the constant case, equal eigenvalues lead with (29) and (30) straightforwardly to the well-known polynomial expressions for the eigenvector components, viz. $w_{i,i+1}(t) = c_{i,i+1}t$ and $w_{i,i+2}(t) = \frac{1}{2}c_{i,i+1}c_{i+1,i+2}t^2 + c_{i,i+2}t$, respectively. Thus, in any case the Jordan form is avoided. In fact, this classic transform turns out to be an anomaly, resulting from the invalid assumption that in any circumstance the mode-amplitudes are constants(*cf.* [14]).
At this point, the dynamic eigenvalue problem (21) with respect to the triangularized system (19) is solved. Combined with (17), a fundamental matrix $\tilde{\mathbf{X}}$ of the original system (1) can now be written as (*cf.* [1])

$$\tilde{\mathbf{X}}(t) = \mathbf{R}(t)\mathbf{W}(t)\exp[\mathbf{\Gamma}(t)]\mathbf{R}^{-1}(0) \quad , \tag{31}$$

where the dynamic eigenvectors $\boldsymbol{w}_j$ $(j = 1, 2, \ldots, n)$ are the respective columns of $\mathbf{W}$, which as a consequence is upper triangular with $\mathbf{W}(0) = \mathbf{I}$ and non-singular, while the diagonal matrix $\boldsymbol{\Gamma}$, denoted as

$$\boldsymbol{\Gamma}(t) = \mathrm{diag}\{\gamma_1(t), \gamma_2(t), \ldots, \gamma_n(t)\} \quad , \tag{32}$$

is given by

$$\boldsymbol{\Gamma}(t) = \int_0^t \boldsymbol{\Lambda}(\tau) d\tau \quad , \tag{33}$$

with

$$\boldsymbol{\Lambda}(t) = \mathrm{diag}\{\lambda_1(t), \lambda_2(t), \ldots, \lambda_n(t)\} \quad . \tag{34}$$

It is stressed that although different particular solutions $\boldsymbol{p}_1$ of the Riccati equation (13) generate in view of (15), (22) and (25) also *different* mode representations, the transition matrix of (1) is uniquely determined.

5. SECOND ORDER SYSTEMS

As is well-known, any scalar second order homogeneous differential equation can be recasted as [15]

$$\ddot{x}_1 + q(t)x_1 = 0 \Longleftrightarrow \dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 \\ -q(t) & 0 \end{bmatrix} \boldsymbol{x} \quad , \tag{35}$$

where the equivalence comes from setting $\dot{x}_1 = x_2$, $\dot{x}_2 = -qx_1$ and $\boldsymbol{x} = [\, x_1 \; x_2 \,]^T$, respectively. Next, system (35) is triangularized by the Riccati transform

$$\boldsymbol{x} = \mathbf{P}(t)\boldsymbol{y} \quad \text{with} \quad \mathbf{P}(t) = \begin{bmatrix} 1 & 0 \\ p(t) & 1 \end{bmatrix} \quad , \tag{36}$$

yielding (*cf.* (14) and (15))

$$\dot{\boldsymbol{y}} = \begin{bmatrix} p(t) & 1 \\ 0 & -p(t) \end{bmatrix} \boldsymbol{y} \quad , \tag{37}$$

provided that $p = p(t)$ is a particular solution of the scalar Riccati differential equation (*cf.* (13))

$$\dot{p} = -p^2 - q \quad . \tag{38}$$

From (37) we obtain the dynamic eigenvalues as (*cf.* (22))

$$\lambda_1(t) = p(t) \quad , \quad \lambda_2(t) = -p(t) \quad . \tag{39}$$

Also, $\mathbf{P}(t)$ in (36) preserves the stability properties iff the norms $\|\mathbf{P}(t)\|$ and $\|\mathbf{P}^{-1}(t)\|$ are bounded (*cf.* [16]). Taking the Euclidean norm, we obtain for any solution p of the Riccati equation (38)

$$\|\mathbf{P}\| = \|\mathbf{P}^{-1}\| = (|p|^2 + 2)^{1/2} \quad . \tag{40}$$

Now, let $q(t) = \omega^2(t) > 0$ and also assume that system (37) is slowly-varying [17]. As a consequence, we may set $\dot{p} \simeq 0$ in (38), and pick $p(t) = j\omega(t)$ with $j^2 = -1$ as a solution of (38). With (39) we then have

$$\lambda_1(t) = j\omega(t) \;,\; \lambda_2(t) = -j\omega(t) \quad , \tag{41}$$

while the corresponding dynamic eigenvectors with respect to system (37) are found as (*cf.* (25))

$$\boldsymbol{v}_1(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \;,\; \boldsymbol{v}_2(t) = \begin{bmatrix} \int_0^t \exp[2\gamma_1(t,\tau)]d\tau \\ 1 \end{bmatrix} \;, \tag{42}$$

where $\gamma_1(t,\tau)$is given by (27), while $\gamma_1(t) = j\omega(t)t$ again because of the slow variation. Then, the transition matrix pertaining to system (35) is obtained as

$$\mathbf{X}(t,t_0) = \begin{bmatrix} c(t,t_0) & -\omega^{-1}(t)s(t,t_0) \\ -\omega(t)s(t,t_0) & c(t,t_0) \end{bmatrix} \quad , \tag{43}$$

in which t_0 is the initial time and where we used the abbreviations $c(t;t_0) = \cos[\omega(t)t - \omega(t_0)t_0]$ and $s(t;t_0) = \sin[\omega(t)t - \omega(t_0)t_0]$, respectively. Note that in general the solution is *not* periodic.

Note also that $\|\mathbf{P}(t)\| = \|\mathbf{P}^{-1}(t)\|$ is bounded as long as $\omega(t)$ is bounded. Hence, $\mathbf{P}(t)$ not only preserves the dynamic eigenvalues but the characteristic Lyapunov exponents as well. Moreover, since $\|\dot{\mathbf{P}}\| = |\dot{p}| = |\dot{\omega}|$ in general is bounded too, it follows that the Riccati transform $\mathbf{P}(t)$ even belongs to the Lyapunov class of transforms as defined in [18].

In the special case $\omega(t) = \omega_0 > 0$ is constant, we have

$$\lambda_1(t) = j\omega_0 \;,\; \lambda_2(t) = -j\omega_0 \quad , \tag{44}$$

leading to solution (43) with $\omega(t)$ replaced by ω_0. Now, it is remarked that the Riccati equation (38) for $\omega = \omega_0$ also admits the alternative solution $p(t) = -\omega_0 \tan(\omega_0 t)$. Hence

$$\lambda_1(t) = -\omega_0 \tan(\omega_0 t) \quad , \quad \lambda_2(t) = \omega_0 \tan(\omega_0 t) \tag{45}$$

is another set of valid dynamic eigenvalues. Instead of (42) with $\omega(t) = \omega_0$, the corresponding dynamic eigenvectors now become

$$\boldsymbol{v}_1(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad , \quad \boldsymbol{v}_2(t) = \begin{bmatrix} (2\omega_0)^{-1}\sin(2\omega_0 t) \\ 1 \end{bmatrix} \quad . \tag{46}$$

However, as can be easily verified, the associated set of mode-vectors lead to the *same* solution (43) with $\omega = \omega_0$. This should be the case because of the afore mentioned uniqueness argument.

Despite an identical system solution for $p = j\omega_0$ and $p = -j\omega_0 \times \tan(\omega_0 t)$, respectively, there is a subtle difference between the corresponding norms of $\mathbf{P}(t)$. Namely, in the first case $\|\mathbf{P}(t)\|$ is with (40) a bounded constant, while in the other case $\|\mathbf{P}(t)\|$ is periodic and bounded, *except* for a countably infinite set of isolated points with measure zero. Since both cases refer to the very same stability behavior, we conclude that the Riccati transform $\mathbf{P}(t)$ with the above mentioned singularity property, will still preserve the characteristic Lyapunov exponents (*cf.* [19]).

Next, consider the modal solutions in steady-state. From the general expression (2) we deduce

$$\boldsymbol{x}(t) \sim \boldsymbol{u}_\infty(t) \exp[\langle\lambda\rangle t] \quad \text{for} \quad t \to \infty \quad , \tag{47}$$

where $\boldsymbol{u}_\infty(t)$ and the constant $\langle\lambda\rangle$ denote the steady-state amplitude and the mean value of $\lambda(t)$, respectively. More precisely, $\langle\lambda\rangle$ is defined as

$$\langle\lambda\rangle = \lim_{t\to\infty} t^{-1} \int_0^t \lambda(\tau) d\tau \quad . \tag{48}$$

Applied to the mode with dynamic eigenvalue $\lambda(t) = \omega_0 \tan(\omega_0 t)$, we obtain from (48) by a suitable contour deformation around the singularity at $\tau = \pi/2$ in the complex τ-plane

$$\langle\lambda\rangle = \frac{\omega_0}{\pi} \int_0^\pi \tan(\tau) d\tau = \pm j\omega_0 \quad , \tag{49}$$

where the plus or minus sign can be chosen, depending on how the singularity is passed around.

Note that (49) delivers just the eigenvalues that we would obtain by setting $\dot{p} = 0$ in the Riccati equation (38).
Note also that from (42) as well as from (46) it follows that $\|\boldsymbol{v}(t)\|$ is bounded, while from (45) and (49) we respectively have Re$[\langle\lambda\rangle] = 0$. Together with the above mentioned stability preserving property of the transform $\mathbf{P}(t)$, we conclude that system (35) with $q(t) = \omega^2(t)$ and $\omega(t) = \omega_0 > 0$ possesses two zero characteristic Lyapunov exponents [19], corresponding with a neutrally stable system.
Finally, if the sign in the system equation (35) is reversed, the Riccati equation (38) changes in

$$\dot{p} = -p^2 + q \quad , \tag{50}$$

which for $q = \omega_0^2$ with $\omega_0 > 0$ is again satisfied by two qualitatively different particular solutions, namely

$$p(t) = \omega_0 \quad \text{and} \quad p(t) = \omega_0 \tanh(\omega_0 t) \quad , \tag{51}$$

each corresponding with a different set of dynamic eigenvalues $\lambda_{1,2}(t) = \pm p(t)$. Now, it can be verified that both sets generate each two characteristic Lyapunov exponents with respective value $\omega_0 > 0$ and $-\omega_0 < 0$, corresponding with a stable and an unstable mode-vector, respectively. Hence, the system is exponentially unstable, as expected.

6. SUMMARY

By allowing time-varying mode-vector solutions, the classical solution procedure for linear time-invariant systems is naturally generalized to LTV systems. The associated dynamic eigenvalue problem is solved by triangularizing the LTV system equations. It generates the dynamic characteristic equation, which essentially equals a collection of lower order (quadratic) Riccati differential equations [1]. Once solved, the dynamic eigenvalues are known and the corresponding dynamic eigenvectors follow straightforwardly from (25). By uniqueness, different sets of linearly independent mode-vectors lead to the same solution.
For a second order LTV system it is explicitly shown that the applied Riccati transform essentially belong to the Lyapunov class. The advantage of having a degree of freedom in choosing a suitable particular solution of the Riccati equation is also demonstrated.

7. REFERENCES

[1] Kloet P.van der and Neerhoff F.L., *Modal Factorization of Time-Varying Models for Nonlinear Circuits by the Riccati Transform*, to be presented at ISCAS 2001, Sidney, Australia.

[2] Ince E.L., *Ordinary Differential Equations*, Dover Publ., Inc., 1956, pp. 133.

[3] Wilkinson J.H., *The Algebraic Eigenvalue Problem*, Oxford University Press, 1978, pp. 30.

[4] Kamen E.W., *The Poles and Zeros of a Linear Time-Varying System*, Lin. Algebra and its Appl., Vol. 98, 1988, pp. 263-289.

[5] Zhu J. and Johnson C.D., *New Results on the Reduction of Linear Time-Varying Dynamical Systems*, SIAM Journ. Control and Opt., Vol. 27, No. 3, May 1989, pp. 476-494.

[6] Zhu J. and Johnson C.D., *Unified Canonical Forms for Matrices Over a Differential Ring*, Lin. Algebra and its Appl., Vol. 147, March 1991, pp. 201-248.

[7] Zhu J., *Series and Parallel D-spectra for Multi-Input-Multi-Output Linear Time-Varying Systems*, Proc. of the 1996 Southeastern Symposium on System Theory, LSU, Baton Rouge, LA, March 1996; IEEE Computer Society Press, Los Alamitos, CA, pp. 125-129.

[8] Zhu J., *PD-Spectral Theory for Multivariable Linear Time-Varying Systems*, Proc. 36th Conf. on Decis. and Control, San Diego, California USA, Dec. 1997, pp. 3908-3913.

[9] Smith D.R., *Decoupling and Order Reduction via the Riccati Transform*, SIAM Review, Vol. 29, No. 1, March 1987, pp. 91-113.

[10] Gantmacher F.R., *The Theory of Matrices*, Vol. two, Chelsea Publ. Co.,1974, pp. 127.

[11] Gantmacher F.R., *The Theory of Matrices*, loc. cit., pp. 123.

[12] Gantmacher F.R., *The Theory of Matrices*, loc. cit., pp. 118.

[13] Kloet P. van der and Neerhoff F.L., *Diagonalization Algorithms for Linear Time-Varying Dynamic Systems*, Int. Journ. of Systems Science, Vol. 31, No. 8, 2000, pp. 1053-1057.

[14] Wu M.-Y., *A New Concept of Eigenvalues and its Applications*, IEEE Trans. on Autom. Control, Vol. 25, 1980, pp. 824-828.

[15] Adrianova L.Ya., *Introduction to Linear Systems of Differential Equations*, American Math. Soc., 1995, pp. 176.

[16] Rugh W.J., *Linear System Theory*, Prentice-Hall, 1991, pp. 93.

[17] Kloet P. van der and Neerhoff F.L., *Behaviour of Dynamic Eigenpairs in Slowly-Varying Systems*, Proc. NDES'99, Ronne, Denmark, July 15-17, 1999, pp. 9-12.

[18] Gantmacher F.R., *The Theory of Matrices*, loc. cit., pp.117.

[19] Adrianova L.Ya., *Introduction to Linear Systems of Differential Equations*, loc. cit., pp. 25.