# MODAL FACTORIZATION OF TIME-VARYING MODELS FOR NONLINEAR CIRCUITS BY THE RICCATI TRANSFORM

*P. van der Kloet and F.L. Neerhoff*

Delft University of Technology
Faculty of Information Technology and Systems
Electronics Research Laboratory
Mekelweg 4, 2628 CD Delft, The Netherlands
email: F.L.Neerhoff@its.tudelft.nl

**ABSTRACT**

In this paper, general linear time-varying systems are addressed. They are considered as small-signal models of nonlinear circuit solutions. The transition matrix is constructed by repeated Riccati transforms. It is shown that each transform factors out a single mode. Key concepts of time-invariant theory are generalized to the time-varying context. They provide an unified mathematical framework, suitable for assessing the local behavior of nonlinear dynamic circuits.

## 1. INTRODUCTION

The dynamic behavior of small departures from a given nominal time-dependent nonlinear circuit solution is described. Besides from other applications, the results are significant for assessing the local stability properties and noise performance of inherently nonlinear operating circuits [1].
Assuming that a local linearization of the (possibly strongly) nonlinear nominal trajectory in state-space is permitted, an associated topologically identical, linear time-varying (LTV) small-signal circuit model may be employed. The nonlinear behavioral characteristics are implied in the associated LTV state-equation [2].
Recently, the LTV approximation was succesfully applied for predicting the local dynamic behavior of class-B amplifiers and dynamic translinear circuits, respectively [3, 4].
In this paper, we present a constructive theory for finding the solution for general LTV systems. It is shown that the solution naturally appears in terms of linearly-independent exponential modes. Each mode is characterized by a time-dependent amplitude and frequency, respectively. Together, they define the extended (LTV) eigenpair associated with an extended eigenvalue problem as introduced by Wu [5, 6]. With these concepts, the author obtained an extended spectral representation of the transition matrix. However, the explicit solution was given as a 'frozen time' algorithm. As we recently proved, it therefore only applies for a restrictive class of LTV systems [7]. Nevertheless, Wu's spectral theorem provides considerable insight in the solution structure of general LTV systems. In fact, it is an extension of the well-known Floquet decomposition for periodic systems [8].
As scalar LTV systems are concerned, Kamen [9] obtained modal solutions by factoring the associated polynomial differential system operator. For the second order case, the factorization was explicitly given in terms of a solution of the Riccati differential equation.
A related approach was followed by Zhu *et al.* [10, 11]. These authors found modal solutions based on a scalar differential operator factorization due to Cauchy-Floquet. Inspired by the classical series and parallel canonical realizations of constant scalar systems, they introduce two types of (interrelated) time-dependent eigenvalues, each type satisfying a kind of characteristic equation. Then, the scalar system is written in vector form with an associated time-dependent Frobenius companion system matrix. However, in transforming this particular system structure to a general one, serious problems were encountered [12, 13].
In contrast, we start right a way with general LTV systems that automatically imply scalar LTV systems as a spacial case. Therefore, our results are more general and transparant.
Our approach is based on the Riccati transform as described in [14]. Essentially, it effectuates an appropriate order reduction and a subsequent decoupling of the original LTV system. As a key application, in Section 2 a single mode is factored out. Then, in Section 3 a recursive scheme is formulated by which the complete modal spectrum is obtained. Finally, it is shown in Section 4 that each mode satisfies a dynamic eigenvalue problem.
At this place it is remarked that we have adopted the adjective *dynamic* to distinguish LTV concepts from their classical (static) counterparts (*cf.* [15, 7]).
Inherent with the Riccati transform, the order reduction demands for a solution of a lower order Riccati differential equation. In Section 4 it is shown that this equation essentially constitutes the dynamic characteristic equation (*cf.* [16]). From it, the dynamic eigenvalues can be computed. On their turn, dynamic eigenvalues are closely related to Lyapunov exponents [17, 18]. That is why the Riccati equation plays a fundamental role in the stability theory of nonlinear systems.

## 2. FACTORING OUT A SINGLE MODE

Consider the $n$-dimensional nonautonomous state-equation

$$\dot{\boldsymbol{x}}_n = \mathbf{A}_n(t)\boldsymbol{x}_n, \tag{1}$$

where $\mathbf{A}_n$ denotes the known system matrix with continuous elements $a_{ij} = a_{ij}(t)$ and $\boldsymbol{x}_n = \boldsymbol{x}_n(t)$ the unknown state-vector, respectively. In order to construct the state transition matrix $\mathbf{X}_n(t, t_0)$ of system (1) with $t_0$ the initial time, the following coordinate transforms are successively applied

$$\boldsymbol{x}_n = \mathbf{P}_n(t)\boldsymbol{y}_n \text{ (a) and } \boldsymbol{y}_n = \mathbf{Q}_n(t)\boldsymbol{z}_n \text{ (b) }, \tag{2}$$

in which $\boldsymbol{y}_n = \boldsymbol{y}_n(t)$ and $\boldsymbol{z}_n = \boldsymbol{z}_n(t)$ are new state-vectors, while the $n$-dimensional (unimodular) matrices $\mathbf{P}_n$ and $\mathbf{Q}_n$ are respectively introduced as [14]

$$\mathbf{P}_n(t) = \begin{bmatrix} \mathbf{I}_{n-1} & \boldsymbol{0} \\ \boldsymbol{p}_{n-1}^T(t) & 1 \end{bmatrix} \text{(a)}, \mathbf{Q}_n(t) = \begin{bmatrix} \mathbf{I}_{n-1} & \boldsymbol{q}_{n-1}(t) \\ \boldsymbol{0}^T & 1 \end{bmatrix} \text{(b)} \tag{3}$$

Here, $\mathbf{I}_{n-1}$ denotes the $(n-1)$ identity matrix while $\boldsymbol{p}_{n-1}$ and $\boldsymbol{q}_{n-1}$ are $(n-1)$ column vectors, respectively ($T$ stands for the transpose).

Next, the system matrix $\mathbf{A}_n$ is partitioned as follows

$$\mathbf{A}_n(t) = \begin{bmatrix} \tilde{\mathbf{A}}_{n-1}(t) & \boldsymbol{a}_{12}(t) \\ \boldsymbol{a}_{21}^T(t) & a_{nn}(t) \end{bmatrix} \quad , \tag{4}$$

where $\tilde{\mathbf{A}}_{n-1}$ denotes the $(n-1)$ left upper square block of $\mathbf{A}_n$, while $\boldsymbol{a}_{12}$ and $\boldsymbol{a}_{21}$ are $(n-1)$ dimensional column vectors. By evoking (2a) we obtain from (1) the following state-equation for $\boldsymbol{y}_n$

$$\dot{\boldsymbol{y}}_n = \begin{bmatrix} [\tilde{\mathbf{A}}_{n-1}(t) + \boldsymbol{a}_{12}(t)\boldsymbol{p}_{n-1}^T(t)] & \boldsymbol{a}_{12}(t) \\ \boldsymbol{0}^T & \lambda_n(t) \end{bmatrix} \boldsymbol{y}_n \quad , \tag{5}$$

provided that $\boldsymbol{p}_{n-1} = \boldsymbol{p}_{n-1}(t)$ is a solution of the $(n-1)$ vector Riccati differential equation with time-varying coefficients

$$\dot{\boldsymbol{p}}_{n-1}^T = - \boldsymbol{p}_{n-1}^T\tilde{\mathbf{A}}_{n-1} + \boldsymbol{a}_{21}^T - \boldsymbol{p}_{n-1}^T\boldsymbol{a}_{12}\boldsymbol{p}_{n-1}^T + a_{nn}\boldsymbol{p}_{n-1}^T \; , \tag{6}$$

while the scalar function $\lambda_n = \lambda_n(t)$ is given by

$$\lambda_n = a_{nn} - \boldsymbol{p}_{n-1}^T\boldsymbol{a}_{12} \quad . \tag{7}$$

Note that *any* solution of the (quadratic) Riccati equation forces the system matrix in (5) to be block triangular.

Similarly, by evoking (2b) system (5) goes into the following block diagonal state-equation for $\boldsymbol{z}_n$

$$\dot{\boldsymbol{z}}_n = \begin{bmatrix} [\tilde{\mathbf{A}}_{n-1}(t) + \boldsymbol{a}_{12}(t)\boldsymbol{p}_{n-1}^T(t)] & \boldsymbol{0} \\ \boldsymbol{0}^T & \lambda_n(t) \end{bmatrix} \boldsymbol{z}_n \quad , \tag{8}$$

provided that $\boldsymbol{q}_{n-1} = \boldsymbol{q}_{n-1}(t)$ is *any* solution of the linear $(n-1)$ vector differential equation with time-varying coefficients

$$\dot{\boldsymbol{q}}_{n-1} = (\tilde{\mathbf{A}}_{n-1} + \boldsymbol{a}_{12}\boldsymbol{p}_{n-1}^T - \lambda_n)\boldsymbol{q}_{n-1} + \boldsymbol{a}_{12} \, . \tag{9}$$

For known solutions $\boldsymbol{p}_{n-1}$ and $\boldsymbol{q}_{n-1}$, the transform matrices $\mathbf{P}_n$ and $\mathbf{Q}_n$ in (2) are respectively known too. The combined transform $\mathbf{P}_n\mathbf{Q}_n$ that carries system (1) into system (8) is called a Riccati transform [14].

As a result, system (8) falls apart into two decoupled lower order state-equations of which the scalar version follows as

$$\dot{z}_{n,n} = \lambda_n(t)z_{n,n} \, , \tag{10}$$

where $z_{n,n} = z_{n,n}(t)$ denotes the $n$-th element of $\boldsymbol{z}_n$. Integration of (10) yields

$$z_{n,n}(t) = z_{n,n}(t_0)\exp[\gamma_n(t,t_0)] \, , \tag{11}$$

where the scalar function $\gamma_n = \gamma_n(t,t_0)$ is given by

$$\gamma_n(t,t_0) = \textstyle\int_{t_0}^{t} \lambda_n(\tau)d\tau \, . \tag{12}$$

Now, by using (2), combined with $\gamma_n(t,t_0) = \gamma_n(t,0) - \gamma_n(t_0,0)$ (*cf.* (12)), it is readily observed that the solution $\boldsymbol{x}_n$ of (1) has the following structure

$$\boldsymbol{x}_n(t) = \tilde{\boldsymbol{x}}_n(t) + \boldsymbol{u}_n(t)\exp[\gamma_n(t,0)] \, . \tag{13}$$

Here, the second term in the right-hand side represents a *mode-vector* with time-dependent *amplitude* $\boldsymbol{u}_n(t)$ and *frequency* $\lambda_n(t) = \dot{\gamma}_n(t,0)$, respectively.

Note that instead of system (1), now the Riccati equation (6) plus the linear time-varying equation (9) have to be solved. Since any *particular* solution for each of these equations suffices, the mode-vector in (13) is *not* uniquely determined (*cf.* [5, 7, 9, 10]). However, the transition matrix $\mathbf{X}_n(t,t_0)$ under construction is [18].

## 3. THE MODAL SPECTRUM

Let the first $(n-1)$ elements of $\boldsymbol{z}_n$ be denoted as the $(n-1)$ column vector $\boldsymbol{x}_{n-1}$, then we deduce from (8) the following state-equation for $\boldsymbol{x}_{n-1} = \boldsymbol{x}_{n-1}(t)$

$$\dot{\boldsymbol{x}}_{n-1} = \mathbf{A}_{n-1}(t)\boldsymbol{x}_{n-1}, \tag{14}$$

where the $(n-1)$ square matrix $\mathbf{A}_{n-1} = [\tilde{\mathbf{A}}_{n-1} + \boldsymbol{a}_{12}\boldsymbol{p}_{n-1}^T]$ is known since $\tilde{\mathbf{A}}_{n-1}$, $\boldsymbol{a}_{12}$ and $\boldsymbol{p}_{n-1}$ are known from (4) and (6), respectively. From now on, we proceed as in the preceeding section, resulting in a state-equation of order $(n-2)$, etc. After $(n-1)$ Riccati transforms, where every transform essentially factors out an extra *orthogonal* mode, we obtain the scalar state-equation $\dot{z}_{1,1} = \lambda_1(t)z_{1,1}$. Then, system (1) is effectively diagonalized and the transition matrix $\mathbf{X}_n(t,t_0)$ is easily found.

In order to formalize this recursive scheme, $\mathbf{X}_n(t,t_0)$ is first expressed in the transition matrix $\mathbf{X}_{n-1}(t,t_0)$ of system (14). Thus, by using (2) and the definition property of the transition matrix, after the first factorization we obtain with (8) and (11)

$$\mathbf{X}_n(t,t_0) = \mathbf{R}_n(t)\begin{bmatrix} \mathbf{X}_{n-1}(t,t_0) & \boldsymbol{0} \\ \boldsymbol{0}^T & \exp[\gamma_n(t,t_0)] \end{bmatrix}\mathbf{R}_n^{-1}(t_0) \, , \tag{15}$$

in which $\mathbf{R}_n = \mathbf{P}_n\mathbf{Q}_n$.

By replacing $n$ by $(n-1)$ in (15), there results after the second factorization

$$\mathbf{X}_{n-1}(t,t_0) = \mathbf{R}_{n-1}(t)\mathbf{D}_{n-1}(t,t_0)\mathbf{R}_{n-1}^{-1}(t_0) \, , \tag{16}$$

where the $(n-1)$ block diagonal matrix $\mathbf{D}_{n-1}$ is given by

$$\mathbf{D}_{n-1}(t,t_0) = \begin{bmatrix} \mathbf{X}_{n-2}(t,t_0) & \boldsymbol{0} \\ \boldsymbol{0}^T & \exp[\gamma_{n-1}(t,t_0)] \end{bmatrix} \, . \tag{17}$$

By introducing the matrices $\mathbf{S}_n^{(1)} = \mathbf{R}_n$ and $\mathbf{S}_n^{(2)} = \mathbf{T}_n = \begin{bmatrix} \mathbf{R}_{n-1} & \boldsymbol{0} \\ \boldsymbol{0}^T & 1 \end{bmatrix}$, we obtain with (16) the identity

$$\begin{bmatrix} \mathbf{X}_{n-1} & \boldsymbol{0} \\ \boldsymbol{0}^T & \exp(\gamma_n) \end{bmatrix} = \mathbf{T}_n(t)\begin{bmatrix} \mathbf{D}_{n-1} & \boldsymbol{0} \\ \boldsymbol{0}^T & \exp(\gamma_n) \end{bmatrix}\mathbf{T}_n^{-1}(t_0) \tag{18}$$

which is subsequently plugged into (15), yielding

$$\mathbf{X}_n(t,t_0) = \mathbf{S}_n(t)\begin{bmatrix} \mathbf{D}_{n-1}(t,t_0) & \boldsymbol{0} \\ \boldsymbol{0}^T & \exp[\gamma_n(t,t_0)] \end{bmatrix}\mathbf{S}_n^{-1}(t_0) \tag{19}$$

in which $\mathbf{S}_n = \mathbf{S}_n^{(1)}\mathbf{S}_n^{(2)}$. Now, by defining the matrices $\mathbf{U}_n = \mathbf{U}_n(t)$ and $\mathbf{S}_n^{(k)} = \mathbf{S}_n^{(k)}(t)$ for $k = 1, 2, \dots, (n-1)$ respectively as

$$\mathbf{U}_n = \prod_{k=1}^{(n-1)} \mathbf{S}_n^{(k)} \, , \mathbf{S}_n^{(k)} = \begin{bmatrix} \mathbf{R}_{n-(k-1)} & \mathbf{0} \\ \mathbf{0}^T & \mathbf{I}_{k-1} \end{bmatrix} , \qquad (20)$$

we finally arrive by mathematical induction at the following expression for the state transition matrix of system (1)

$$\mathbf{X}_n(t, t_0) = \mathbf{U}_n(t) \exp[\mathbf{\Gamma}_n(t, t_0)] \mathbf{U}_n^{-1}(t_0) \, , \qquad (21)$$

where the diagonal matrix $\mathbf{\Gamma}_n$, denoted as

$$\mathbf{\Gamma}_n(t, t_0) = \text{diag}\{\gamma_1(t, t_0), \gamma_2(t, t_0), \dots, \gamma_n(t, t_0)\} \, , \qquad (22)$$

is given by

$$\mathbf{\Gamma}_n(t, t_0) = \textstyle\int_{t_0}^{t} \mathbf{\Lambda}_n(\tau) d\tau \qquad (23)$$

with

$$\mathbf{\Lambda}_n(t) = \text{diag}\{\lambda_1(t), \lambda_2(t), \dots, \lambda_n(t)\} \, . \qquad (24)$$

Now, let $\tilde{\mathbf{X}}_n(t)$ be a fundamental matrix of system (1), thus $\mathbf{X}_n(t, t_0) = \tilde{\mathbf{X}}_n(t)\tilde{\mathbf{X}}_n^{-1}(t_0)$, then we obtain with (23) from (21)

$$\tilde{\mathbf{X}}_n(t) = \mathbf{U}_n(t) \exp[\mathbf{\Gamma}_n(t, 0)] \, . \qquad (25)$$

Furthermore, let the $n$-vectors $\boldsymbol{x}_n^{(i)}$ and $\boldsymbol{u}_n^{(i)}$ $(i = 1, 2, \dots, n)$ denote the columns of $\tilde{\mathbf{X}}_n$ and $\mathbf{U}_n$ respectively, then we observe from (25)

$$\boldsymbol{x}_n^{(i)}(t) = \boldsymbol{u}_n^{(i)}(t) \exp[\gamma_i(t, 0)] \, . \qquad (26)$$

In conclusion, the $n$ columns of $\tilde{\mathbf{X}}_n$ represent $n$ mode-vectors with time-dependent amplitudes $\boldsymbol{u}_n^{(i)}(t)$ and frequencies $\lambda_i(t) = \dot{\gamma}_i(t, 0)$, respectively. Therefore, expression (21) represents the modal spectrum pertaining to general LTV systems. This is in complete agreement with Wu's spectral theorem [6]. Note however, that we obtained (21) by its *de facto* construction with repeated Riccati transforms.

## 4. THE DYNAMIC EIGENVALUE PROBLEM

Since the columns of $\tilde{\mathbf{X}}_n$ by definition satisfy state-equation (1), the following equation for the mode quantities $\boldsymbol{u}_n^{(i)}$ and $\lambda_i$ holds (*cf.* [6, 10, 18])

$$[\mathbf{A}_n(t) - \lambda_i(t)\mathbf{I}_n] \, \boldsymbol{u}_n^{(i)}(t) = \dot{\boldsymbol{u}}_n^{(i)}(t) \, . \qquad (27)$$

For a constant amplitude $\boldsymbol{u}_n^{(i)}$, this equation obviously reduces to the classical eigenvalue problem. Moreover, as in the constant case, it turns out that $\lambda_i(t)$ is invariant under a coordinate transformation [5, 7, 18]. Therefore, equation (27) is called the *dynamic* eigenvalue problem pertaining to system (1). In this respect, $\boldsymbol{u}_n^{(i)}(t)$ and $\lambda_i(t)$ are called *dynamic* eigenvectors and *dynamic* eigenvalues, respectively.
The dynamic eigenvalue $\lambda_n$ as defined by (7) follows from a solution of the lower order Riccati equation (6). Substitution of (7) in (6) yields

$$\dot{\boldsymbol{p}}_{n-1}^T = \boldsymbol{p}_{n-1}^T(\lambda_n \mathbf{I}_{n-1} - \tilde{\mathbf{A}}_{n-1}) + \boldsymbol{a}_{21}^T \, . \qquad (28)$$

If this expression is augmented with (7) there results (*cf.* [16])

$$\dot{\boldsymbol{v}}_n^T = \boldsymbol{v}_n^T \begin{bmatrix} (\lambda_n \mathbf{I}_{n-1} - \tilde{\mathbf{A}}_{n-1}) & -\boldsymbol{a}_{12} \\ -\boldsymbol{a}_{21}^T & (\lambda_n - a_{nn}) \end{bmatrix} , \qquad (29)$$

where the row vector $\boldsymbol{v}_n^T = \boldsymbol{v}_n^T(t)$ is given by $\boldsymbol{v}_n^T = [\boldsymbol{p}_{n-1}^T - 1]$. In view of the displayed partitioning in (4), equation (29) can be put together as

$$\dot{\boldsymbol{v}}_n^T(t) = \boldsymbol{v}_n^T(t)[\lambda_n(t)\mathbf{I}_n - \mathbf{A}_n(t)] \, , \qquad (30)$$

where $\mathbf{I}_n$ denotes the $n$-dimensional identity matrix.
For the constant case, the Riccati equation (6) admits a constant solution for $\boldsymbol{p}_{n-1}^T$. Then, $\dot{\boldsymbol{v}}_n = \boldsymbol{0}$ but $\boldsymbol{v}_n \neq \boldsymbol{0}$. As a consequence, equation (30) reduces to a homogeneous linear equation. It has non-zero solutions iff $\lambda_n$ satisfies the classical characteristic equation

$$\det[\mathbf{A}_n - \lambda \mathbf{I}_n] = 0 \, . \qquad (31)$$

For this reason, the collection of $(n-1)$ augmented (quadratic) Riccati differential equations that generates the complete set of dynamic eigenvalues, is called the *dynamic* characteristic equation associated with system (1). From it, the dynamic eigenvalues can be computed.

## 5. SUMMARY

The transition matrix for a general LTV system is constructed by repeated Riccati transforms. Each transform factors out a single mode, that on its turn satisfies a dynamic eigenvalue problem.
A collection of augmented lower order (quadratic) Riccati differential equations is recognized as the associated dynamic characteristic equation. As in the classical context where an algebraic polynomial equation has to be solved, a solution of its dynamic counterpart is not *per se* easy to obtain [19]. However, its theoretical significance is obvious. It generates the dynamic eigenvalues that are closely related with Lyapunov exponents. The latter determine the local stability of the associated nonlinear dynamic system solution.

## 6. REFERENCES


[1] Stern T.E., *Theory of Nonlinear Networks and Systems, an Introduction*, Addison-Wesley, 1965, pp. 234.

[2] Neerhoff F.L., Kloet P. van der, Staveren A. van and Verhoeven C.J.M., *Time-Varying Small-Signal Circuits for Nonlinear Electronics*, Proc. NDES 2000, Catania, Italy, May 18-20, 2000, pp. 85-89.

[3] Staveren A. van, Cordenier T.H.A.J., Kuijstermans F.C.M., Kloet P. van der, Neerhoff F.L., Verhoeven C.J.M. and Roermond A.H.M. van, *The Linear Time-Varying Approach applied to the Design of a Negative-Feedback Class-B Output Amplifier*, Proc. ISCAS, Orlando, USA, June 1999, pp. II-204-207.

[4] Diepstraten F.M., Kuijstermans F.C.M., Serdijn W.A., Kloet P. van der, Staveren A. van, Neerhoff F.L., Verhoeven C.J.M. and Roermond A.H.M. van, *Dynamic Behaviour of Dynamic Translinear Circuits: the Time-Varying Approach*, submitted to IEEE Trans. on Circuits and Systems.

[5] Wu M.-Y., *A New Concept of Eigenvalues and its Applications*, IEEE Trans. on Autom. Control, Vol. 25, 1980, pp. 824-828.

[6] Wu M.-Y., *On the Stability of Linear Time-Varying Systems*, Int. Journ. of Systems Science, Vol. 15, 1984, pp. 137-150.

[7] Kloet P. van der and Neerhoff F.L., *Diagonalization Algorithms for Linear Time-Varying Dynamic Systems*, Int. Journ. of Systems Science, Vol. 31, No. 8, 2000, pp. 1053-1057.

[8] Rugh W.J., *Linear System Theory*, Prentice-Hall, 1991, pp. 73.

[9] Kamen E.W., *The Poles and Zeros of a Linear Time-Varying System*, Lin. Algebra and its Appl., Vol. 98, 1988, pp. 263-289.

[10] Zhu J. and Johnson C.D., *New Results on the Reduction of Linear Time-Varying Dynamical Systems*, SIAM Journ. Control and Opt., Vol. 27, No. 3, May 1989, pp. 476-494.

[11] Zhu J. and Johnson C.D., *Unified Canonical Forms for Matrices Over a Differential Ring*, Lin. Algebra and its Appl., Vol. 147, March 1991, pp. 201-248.

[12] Zhu J., *Series and Parallel D-spectra for Multi-Input-Multi-Output Linear Time-Varying Systems*, Proc. of the 1996 Southeastern Symposium on System Theory, LSU, Baton Rouge, LA, March 1996; IEEE Computer Society Press, Los Alamitos, CA, pp. 125-129.

[13] Zhu J., *PD-Spectral Theory for Multivariable Linear Time-Varying Systems*, Proc. 36th Conf. on Decis. and Control, San Diego, California USA, Dec. 1997, pp. 3908-3913.

[14] Smith D.R., *Decoupling and Order Reduction via the Riccati Transform*, SIAM Review, Vol. 29, No. 1, March 1987, pp. 91-113.

[15] Kloet P. van der and Neerhoff F.L., *On Eigenvalues and Poles for Second Order Linear Time-Varying Systems*, Proc. NDES'97, Moscow, Russia, June 26-27, 1997, pp. 300-305.

[16] Kloet P. van der, Kuijstermans F.C.M., Neerhoff F.L., Staveren A. van and Verhoeven C.J.M., *A note on Dynamic Eigenvalues and Slowly-Varying Systems*, Proc. X. Int. Symp. Theor. Electr. Eng., Magdeburg, Gemany, 1999, pp. 141-144.

[17] Neerhoff F.L. and Kloet P. van der, *Frequency Behaviour of Time-Varying Small-Signal Models of Nonlinear Circuits*, Proc. NDES 2000, Catania, Italy, May 18-20, 2000, pp. 85-89.

[18] Neerhoff F.L. and Kloet P. van der, *A Complementary View on Time-Varying Systems*, to be presented at ISCAS 2001, Sydney, Australia.

[19] Bittanti S., Laub A.J. and Willems J. C (Eds), *The Riccati Equation*, Springer-Verlag, 1991.