FREQUENCY BEHAVIOUR OF TIME-VARYING SMALL-SIGNAL MODELS OF NONLINEAR CIRCUITS

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Abstract— *The frequency behaviour of time-varying small-signal models of nonlinear circuits is described by a modal expansion. Key concepts of time-invariant theory as normal mode, natural frequency and pole are generalized into the time-varying context. It is shown that with the exception of slowly-varying circuits, the accumulated meaning of a pole in time-invariant theory is lost for time-varying systems. Their impact turns out to be restricted exclusively to the high-frequency behaviour. Moreover, these high-frequency poles no longer coincide with the natural frequencies, while their location in the right-hand side of the complex frequency-plane neither predicts instability. Instead, their classic role in stability theory has to be replaced by a newly introduced complex Lyapunov exponent. This concept is closely related to the earlier introduced dynamic eigenvalue. The latter formally follows from a Riccati differential equation in its formal meaning of the generalized characteristic equation pertaining to linear time-varying systems. Both the complex Lyapunov exponent and the dynamic eigenvalue define to some extent a generalized pole concept.*

I. INTRODUCTION

The frequency behaviour of relatively small variations of nonlinear dynamic circuit solutions is described. Small-signal model circuits share their topology with the associated original nonlinear circuits, while each nonlinear circuit element is replaced by an incremental one, evaluated along the nonlinear solution. As a consequence, they are linear time-varying (LTV) by nature [1], [2].

As concerning the analysis of the frequency behaviour of LTV systems, a great deal of effort is put in the development of an extended LTV transferfunction concept that generalizes the well-known linear time-invariant (LTI) transferfunction [4] - [7]. Except from [7], where operator theory is used, these contributions are all based on a direct frequency approach. In this paper, we describe the frequency behaviour from a modal expansion point of view [8]. With a single mode circuit as a starting point, key concepts of time-invariant theory as normal mode, natural frequency and pole are generalized to the time-varying context. They are all associated with the earlier introduced dynamic eigenvalues [9] - [13].

In section 2 a first order LTV RC-circuit is first analyzed in the time-domain. To that aim, the state equation for the electrical charge is formulated and subsequently solved by staightforward integration. The solution is interpreted as a normal mode with constant amplitude and time dependent phase, respectively. The latter is associated with a time dependent natural frequency as the physical interpretation of a dynamic eigenvalue. Here, a time dependent frequency has to be understood in the sense of [14].

In section 3, the frequency spectra for the circuit variables are easily obtained by taking the Fourier transform of their modal representations. It turns out that their high-frequency spectra are determined by a real pole in the complex frequency-plane. Next, it is shown that on condition of a negative Lyapunov exponent, the LTV circuit solution is even stable if the high-frequency pole is located in the right-half complex frequency-plane.

In section 4, the state-equation with respect to a general source-free LTV circuit is first diagonalized by a suitable chosen Lyapunov transformation. As shown earlier, the associated Lyapunov matrix is formally obtained by solving a Riccati differential equation. The latter turns out to be the generalized characteristic equation for LTV systems [10], [13]. From it, the dynamic eigenvalues follow immediately. In general, they are complex valued and not necessarily two-by-two complex conjugated [9]. Once the system is diagonalized, the solutions then follow by straightforward integration. They appear as uncoupled normal modes with the dynamic eigenvalues as time dependent natural frequencies. With them, the complete solution of the system equations is written as a linear combination of oscillatory modes. Each constituent mode is characterized by a time dependent complex amplitude vector and a time dependent complex natural frequency, respectively. Thus, in stead of first formulating the dynamic eigenvalue problem as in [8], [11], [12], the modal expansion now follows in a straightforward way.

From now on, we proceed as in the preceeding section. Therefore, essentially the same conclusions are reached concerning the frequency spectrum pertaining to any individual mode. Finally, a complex Lyapunov exponent is introduced as the asymptotic mean value of a dynamic eigenvalue. It replaces the role of a pole in time-invariant stability analysis. In this respect, both the complex Lyapunov exponent and the dynamic eigenvalue define to some extent a generalized pole concept.

II. First order LTV circuit in time-domain

Consider an elementary LTV RC-combination where both elements are time-varying. Let the capacitor be characterized by a time-dependent elastance $s = s(t)$ [F^{-1}]. Then, its constitutive relation is given by

$$u(t) = s(t)q(t) \;, \tag{1}$$

where $u = u(t)$ and $q = q(t)$ denote the voltage across the RC-combination and the accumulated charge by the current $i = i(t)$ through it, respectively. In view of the Kirchhoff laws, we then have for the resistor

$$i(t) = -g(t)u(t) \;, \tag{2}$$

in which the time-dependent conductance is denoted by $g = g(t)$ [S]. Combining (2) with (1) yields

$$i(t) = -g(t)s(t)q(t) \;. \tag{3}$$

Since $i = \dot{q}(t)$, where the dot refers to the time derivative, we arrive with (3) at the following homogeneous linear differential equation with time-dependent coefficient

$$\dot{q}(t) = \lambda(t)q(t) \;, \tag{4}$$

where $\lambda(t)$ denotes the earlier introduced dynamic eigenvalue with respect to (4) [9], here given by

$$\lambda(t) = -g(t)s(t) \;. \tag{5}$$

With the assumption that $q(0)$ is a known quantity, it is easy to verify that

$$q(t) = q(0)\exp(\gamma(t)) \tag{6}$$

with

$$\gamma(t) = \int_0^t \lambda(\tau)d\tau \;, \tag{7}$$

is the solution of (4) for $t > 0$. Now, the voltage $u(t)$ and the current $i(t)$ follows respectively from (1) and (3) as

$$u(t) = q(0)s(t)\exp(\gamma(t)) \tag{8}$$

and

$$i(t) = q(0)\lambda(t)\exp(\gamma(t)) \;. \tag{9}$$

The expressions (8) and (9) are subsequently interpreted as an oscillatory mode with time dependent amplitude $q(0)s(t)$ and $q(0)\lambda(t)$, respectively, and time dependent phase $\gamma(t)$. Moreover, we observe from (7) that $\dot{\gamma}(t) = \lambda(t)$ has the meaning of an inverse time dependent relaxation time ('time-constant') or, more generally, a time dependent natural frequency.

Finally, it is remarked that the electrical charge $q(t)$ is the only state variable for which in absence of a current impulse, time continuity is ensured (compare [15]). As a consequence, the state-equation (4) leads exclusively to that modal solution (6) for which the amplitude is a constant, while its phase $\gamma(t)$ is unambiguously defined. For this reason, the modal representation (6) is called a normal mode [16]. The other circuit variables $u(t)$ and $i(t)$ follow from $q(t)$ upon multiplication by a time dependent circuit parameter, and therefore share the same phase $\gamma(t)$ in the modal representations (8) and (9), respectively. In conclusion, $\lambda(t)$ is not only the dynamic eigenvalue with respect to state-equation (4) but for the LTV RC-combination as a system as well.

III. Frequency spectrum of first-order LTV circuit

Due to the results of the preceeding section, the frequency spectrum $U(\omega)$ of the voltage $u(t)$ across the LTV RC-combination is in view of causality easily found to be

$$U(\omega) = q(0)\int_0^\infty s(t)\exp(\gamma(t) - j\omega t)dt \;, \tag{10}$$

and analogous for the frequency spectrum $I(\omega)$ of the current $i(t)$ through it as

$$I(\omega) = q(0)\int_0^\infty \lambda(t)\exp(\gamma(t) - j\omega t)dt \;. \tag{11}$$

We check these expressions for a LTI RC-circuit. With $s(t) = C^{-1}$, where C denotes the capacitance, and $g(t) = R^{-1}$, we successively have $\lambda(t) =$

$-(RC)^{-1} = \lambda$ and $\gamma(t) = \lambda t$. Assuming stability, thus $\lambda < 0$, the integrals in (10) and (11) then yield

$$U(\omega) = \frac{-Rq(0)\lambda}{(j\omega - \lambda)} \quad \text{and} \quad I(\omega) = \frac{q(0)\lambda}{(j\omega - \lambda)} \,. \quad (12)$$

It follows that the frequency spectrum is completely determined by a single pole for $j\omega = \lambda$ in the complex $j\omega$-plane. Moreover, the pole coincides with the constant natural frequency λ.

We now return to the time-varying context and try to work the integrals (10) and (11) towards the results (12). With the substitution (compare (7))

$$dt = (\lambda(t) - j\omega)^{-1} d(\gamma(t) - j\omega t) \,, \qquad (13)$$

the integral in (11) can be converted into the following expression

$$I(\omega) = \frac{q(0)\lambda(0)}{(j\omega - \lambda(0))} + \\ + q(0) \int_0^\infty \frac{j\omega \dot{\lambda}(t)}{(\lambda(t) - j\omega)^2} \exp(\gamma(t) - j\omega t) dt \,, \qquad (14)$$

while (10) delivers a similar result. Thereby, it is assumed that 1. $\lambda(t)$ is bounded for $t \to \infty$ and 2. $t^{-1}\gamma(t) < 0$ for $t \to \infty$. Thus, the LTV RC-circuit is assumed to be stable and, as a consequence, is characterized by a negative Lyapunov-exponent χ. The latter is defined as [17]

$$\chi = \lim_{t\to\infty} t^{-1} \ln |x(t)| \,, \qquad (15)$$

where $x(t)$ equals either $u(t)$ or $i(t)$ as they are given by (8) and (9), respectively. It follows that under the asserted assumption of a bounded real value of $\lambda(t)$, the stability condition $\chi < 0$ can also be written as

$$\lim_{t\to\infty} t^{-1} \int_0^t \lambda(\tau) d\tau < 0 \,, \qquad (16)$$

from which it is clearly observed that not only negative values of $\lambda(t)$ are allowed, but possitive values as well, provided that on the long run the mean value of $\lambda(t)$ turns out to be negative.

By repeated use of substitution (13) a series expansion results that due to the theorem of Riemann-Lebesgue, represents a progressively accurate approximation for the high-frequency spectrum. In particular, it follows

$$I(\omega) = \frac{q(0)\lambda(0)}{(j\omega - \lambda(0))} + \mathcal{O}(\omega^{-2}) \,\text{for}\, \omega \to \infty \,, \quad (17)$$

while it is analogously found

$$U(\omega) = -\frac{R(0)q(0)\lambda(0)}{(j\omega - \lambda(0))} + \mathcal{O}(\omega^{-2}) \,\text{for}\, \omega \to \infty \,, \qquad (18)$$

in which $R(0) = (g(0))^{-1}$. Note that (17) and (18) are in agreement with the initial value theorem for Fourier integrals of causal functions.

From (17) and (18) it is observed that the high-frequency spectrum of a linear time-varying RC-circuit is dominated by a single pole at $j\omega = \lambda(0)$ in the complex $j\omega$-plane. Different from the time-invariant RC-combination, the fixed pole not longer coincides with the natural frequency $\lambda(t)$. Moreover, it might be that $\lambda(0) > 0$ while yet the stability condition (16) is satisfied.

If $\dot{\lambda}(t) = 0$, it is easily deduced that

$$I(\omega) = \frac{q(0)\lambda(0)}{(j\omega - \lambda(0))} \,\text{and}\, U(\omega) = -\frac{R(0)q(0)\lambda(0)}{(j\omega - \lambda(0))} \,. \qquad (19)$$

Since the condition $\dot{\lambda}(t) \simeq 0$ applies for a slowly-varying RC-circuit [10], we conclude that the frequency spectrum, just as for its time-invariant antipode, is completely determined by a single pole for $j\omega = \lambda(0)$. But different from the time-invariant case, the slowly-varying RC-circuit is even stable if $\lambda(0) > 0$, provided that $\lambda(t)$ keeps satisfying (16).

Finally, it is remarked that due to the duality principle in circuit theory, a LTV RL-combination yields identical results.

IV. Frequency spectrum by modal expansion

Consider a general source-free LTV electrical circuit. Then, the state equations can be written as (see [18] for writing state equations)

$$\dot{\boldsymbol{x}} = \mathbf{A}(t)\,\boldsymbol{x} \quad \text{(a) and} \quad \boldsymbol{y}(t) = \mathbf{B}(t)\boldsymbol{x}(t) \quad \text{(b)} \,. \qquad (20)$$

Here, the state n-vector $\boldsymbol{x}(t)$ collects the capacitor charges and inductor fluxes as independent state variables, respectively, while $\mathbf{A}(t)$ is a bounded square matrix of order n. In addition, the m-vector $\boldsymbol{y}(t)$ contains the desired voltages and currents, while the $m \times n$ matrix $\mathbf{B}(t)$ is assumed to be bounded.

In order to obtain a modal expansion for the unknown circuit variable $\boldsymbol{y}(t)$, we first apply a Lyapunov transformation $\mathbf{L}(t)$ that diagonalizes the system equations (20.a) [8]. Thus, by setting

$$\boldsymbol{x}(t) = \mathbf{L}(t)\boldsymbol{z}(t) \,, \qquad (21)$$

system (20.a) goes into the diagonal LTV system

$$\dot{\boldsymbol{z}} = \boldsymbol{\Lambda}(t)\boldsymbol{z} \quad , \tag{22}$$

where the diagonal matrix $\boldsymbol{\Lambda}$ is obtained as [19], [20]

$$\boldsymbol{\Lambda} = \mathbf{L}^{-1}\mathbf{A}\mathbf{L} - \mathbf{L}^{-1}\dot{\mathbf{L}} \quad . \tag{23}$$

As outlined in [9], [10], [13], the particular Lyapunov matrix $\mathbf{L}$ that indeed diagonalizes system (20.a) is found by solving a Riccati differential equation of order $(n-1)$. From a fundamental point of view, this equation turns out to be the generalized characteristic equation with respect to LTV systems [10], [13]. From a practical point of view, the diagonalizing matrix $\mathbf{L}$ may be found by using the iterative procedures as proposed in [8], [11], [12], [13]. Once $\mathbf{L}$ is known, the diagonal matrix $\boldsymbol{\Lambda}$ follows from (23) as

$$\boldsymbol{\Lambda}(t) = \text{diag}\{\lambda_1(t), \lambda_2(t), \ldots, \lambda_n(t)\} \; , \tag{24}$$

in which the elements $\lambda_i(t)$ $(i = 1, 2, \ldots, n)$ are recognized as the dynamical eigenvalues with respect to the dynamic similar systems (22) and (20.a), respectively [9], [11], [12], [13]. In general, they are complex valued and not necessarily two-by-two complex conjugated as they are in the time-invariant case [9].
Next, the diagonal system (22) is solved by straightforward integration. The n independent solutions $\boldsymbol{z}_i(t) (i = 1, 2, \ldots, n)$ are found as

$$\boldsymbol{z}_i(t) = \boldsymbol{z}_i(0) \exp(\gamma_i(t)) \; , \tag{25}$$

in which $\gamma_i(t)$ is given by (compare (7))

$$\gamma_i(t) = \int_0^t \lambda_i(\tau) d\tau \; . \tag{26}$$

As in section 2, the expression (25) is interpreted for each i as an individual normal mode with time dependent complex phase $\gamma_i(t)$ and time dependent complex natural frequency $\lambda_i(t)$, respectively. Since the n modes (25) are completely uncoupled, they form a base in state-space. As a consequence, the solution of (22) can be written as the modal expansion

$$\boldsymbol{z}(t) = \sum_{i=1}^{n} \boldsymbol{z}_i(t) \; , \tag{27}$$

where on account of (21) we have $\boldsymbol{z}(0) = \mathbf{L}^{-1}(0)\boldsymbol{x}(0)$ with $\boldsymbol{x}(0)$ a given initial state. Upon combining (27) and(21) with (20.b) we finally arrive at the desired modal expansion for $\boldsymbol{y}(t)$, namely

$$\boldsymbol{y}(t) = \sum_{i=1}^{n} \boldsymbol{u}_i(t) \exp(\gamma_i(t)) \; , \tag{28}$$

in which the time dependent amplitude vectors $\boldsymbol{u}_i(t)$ are given by $\boldsymbol{u}_i(t) = \mathbf{B}(t)\mathbf{L}(t)\boldsymbol{z}_i(0)$. Note that due to the boundedness of the circuit matrix $\mathbf{B}(t)$ and the Lyapunov matrix $\mathbf{L}(t)$, respectively, the amplitudes $\boldsymbol{u}_i(t)$ are bounded, too.
We now are in he position to proceed in conformity with section 3. Thus, the frequency spectrum of the collected voltages and currents in $\boldsymbol{y}(t)$ are given by the complex valued m-vector $\boldsymbol{Y}(\omega)$ as

$$\boldsymbol{Y}(\omega) = \sum_{i=1}^{n} \int_0^{\infty} \boldsymbol{u}_i(t) \exp(\gamma_i(t) - j\omega t) dt \; . \tag{29}$$

Through substitution (13), we easily obtain in analogy to (17) and (18) for the high-frequency spectrum

$$\boldsymbol{Y}(\omega) = \sum_{i=1}^{n} \frac{\boldsymbol{u}_i(0)}{(j\omega - \lambda_i(0))} + \mathcal{O}(\omega^{-2}) \text{ for } \omega \to \infty \; , \tag{30}$$

while for slowly varying LTV circuits, as in (19), the first term in the right hand-side of (30) applies for any ω. Here, again stability is assumed, which due to the boundedness of the amplitudes $\boldsymbol{u}_i(t)$ comes down to the replacement of (16) by the condition [10], [21].

$$\lim_{t\to\infty} \Re\{t^{-1} \int_0^t \lambda_i(\tau) d\tau\} < 0 \text{ for } i = 1, 2, \ldots, n \; , \tag{31}$$

where $\Re$ denotes the real part. Thus, the Lyapunov exponents associated with the individual modes have all to be negative. In summary, the conclusions pertaining to a single mode circuit, essentially applies to a general LTV circuit as well.
As a final observation, it is remarked that the complex Lyapunov exponents L_i , introduced by the definition

$$L_i = \lim_{t\to\infty} t^{-1} \int_0^t \lambda_i(\tau) d\tau \quad (i = 1, 2, \ldots, n) \; , \tag{32}$$

obviously replace the role of poles in the stability analysis pertaining to time-invariant systems. Note also that the complex Lyapunov exponents L_i are global quantities, contrary to the dynamic eigenvalues λ_i which are locally defined. Moreover, as we proved earlier [9], the dynamic eigenvalues equal the 'right poles' as introduced in [22]. Therefore, both the complex Lyapunov exponents and the dynamic eigenvalues define to some extent a generalized pole concept.

V. Conclusions

Due to linearity, the complete solution of a source-free time-varying circuit can be written as a linear combination of elementary modal solutions, each characterized by a dynamic eigenvector and a dynamic eigenvalue. These mathematical concepts are physically interpreted as a time dependent amplitude and an also time dependent natural frequency, respectively.

The high-frequency behaviour of each constituent mode is dominated by a single, fixed pole. Provided that the associated Lyapunov exponent of a mode is negative, the mode is even stable if its high-frequency pole is located in the right-half complex frequency plane. The frequency behaviour of a slowly-varying mode is completely determined by a single pole. Here again, the location of the pole is not decisive for the stability of a mode.

A complex Lyapunov exponent is introduced as the asymptotic mean value of a dynamic eigenvalue. It is the generalization of a classic pole in the sense that stability is ensured if its real part is negative. On the other hand, the dynamic eigenvalues, to which the complex Lyapunov exponents are closely related, follow from a generalized characteristic equation and equal the 'right poles' as introduced in [22]. In this respect, a (time dependent) dynamic eigenvalue also defines a generalized pole concept. Both the complex Lyapunov exponent and the dynamic eigenvalue coincide with a fixed pole in the time-invariant framework.

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