TIME-VARYING SMALL-SIGNAL CIRCUITS FOR NONLINEAR ELECTRONICS

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Abstract—*Given a nonlinear electronic circuit, an associated linear time-varying small-signal circuit is formally derived by the tableau-method. It has the same topology as the original circuit while each original circuit element is replaced by an incremental one, evaluated along the signal-dependent nonlinear circuit solution. Since the variational circuit is linear in the first place, the designer is now invited to use the results of linear circuit theory. Furthermore, we present time-discrete companion models for linear time-varying elements. By them, the small-signal circuit equations can be efficiently solved numerically.*

I. INTRODUCTION

Linear time-varying (LTV) models are for example important for studying the stability and noise behaviour of nonlinear circuits [1]. Recently, the LTV-approach was applied for nonlinear electronic circuit design [2] - [5]. Thereby, first the nonlinear state-equations had to be formulated and subsequently solved in order to obtain the signal dependent bias trajectory in state-space. The Jacobian along the solution trajectory constitutes the time-varying system matrix of the associated variational LTV state-equations. From them, the dynamic small-signal behavior was deduced by using the concept of dynamical eigenvalues [6] - [10].

However, such a system-oriented approach completely obscures the underlying circuit topology. Therefore, the electronic designer does not fully exploits his expertise with linear design methodologies.

In this paper a complementary, *circuit*-oriented approach is proposed. To that aim, we derive a LTV variational circuit, associated with the original one. Since the resulting small-signal circuit is *linear* in the first place, the designer is now invited to use the results of linear circuit theory. In particular, the superposition principle holds. As a consequence, one may use for example the Thévenin and Norton theorems for ports, including those ports associated with LTV-dynamic elements [11]. Also, linear two-port theory becomes available, together with a variety of inspection techniques that simplify the circuit topology.

In the next section, the associated LTV variational circuit is formally derived by using the tableau-method [12]. In this method, Kirchhoff's voltage law (KVL) is effectuated by writing each branch voltage as a difference of two node-to-datum voltages, while Kirchhoff's current law (KCL) is required for any node, except the datum node, respectively. The branch voltages and branch currents, together with the node-to-datum voltages are all taken as unknown variables. Then, the (linear) Kirchhoff-equations and the (possibly nonlinear) terminal relations of the constituent circuit elements are collected as subsets of equations in the resulting tableau, respectively. Thus, thanks to the introduction of the node-to-datum voltages as additional unknowns, the Kirchhoff-equations and the terminal relations appear as seperated sets of equations.

Now, it becomes completely transparant that the LTV variational circuit has the same topology as the original nonlinear circuit, while each original circuit element is replaced by an incremental one, evaluated along the source-dependent nonlinear circuit solution. As a matter of fact, our argumentation is just a generalization of the derivation of the linear time-invariant variational circuit in [12].

Although the LTV small-signal circuit *equations* were obtained earlier by a different, by far less transparant method, no attempt was made to translate these equations into a real circuit *topology* [13]. Finally, in section 3 we present time-discrete companion models for the LTV circuit elements. By them, the LTV circuit problem can be efficiently solved numerically by standard routines.

II. LTV SMALL-SIGNAL CIRCUIT

Given a nonlinear electronic circuit $\mathcal{C}$ with a directed graph composed of n nodes and b branches. One arbitrarily node is selected as datum node. The b branch currents and b branch voltages are respectively collected in the b-vectors $\boldsymbol{i}$ and $\boldsymbol{u}$ as follows

$$\boldsymbol{i} = [i_1\, i_2 \ldots i_b]^T \text{ and } \boldsymbol{u} = [u_1\, u_2 \ldots u_b]^T\,, \qquad (1)$$

where T denotes the transpose, while the $(n-1)$ node-to-datum voltages go into the $(n-1)$-vector $\boldsymbol{v}$, thus

$$\boldsymbol{v} = [v_1\, v_2 \ldots v_{n-1}]^T\,.$$

The Kirchhoff-equations of $\mathcal{C}$ can now be put into the following linear matrix form

$$\mathbf{A}\boldsymbol{i} = \boldsymbol{0}\ \ (KCL) \text{ and } \boldsymbol{u} - \mathbf{A}^T\boldsymbol{v} = \boldsymbol{0}\ \ (KVL)\,, \quad (2)$$

where $\mathbf{A}$ denotes the reduced node-branch incidence matrix of dimension $(n-1)\times b$. Since $\mathbf{A}$ is of full rank $(n-1)$, we have $(n-1)$ and b linear independent KCL's and KVL's, respectively, making a total of $(n-1)+b$ linear independent Kirchhoff-equations. Next to the Kirchhoff-equations we also have the terminal relations of the constituent circuit elements of $\mathcal{C}$. If we restrict ourselves for the moment to 2-termimal elements (later on we get rid of this restriction), we distinguish the following types of nonlinear elements: 1. independent voltage sources, 2. independent current sources, 3. current controlled resistors, 4. voltage controlled resistors, 5. voltage controlled capacitors, 6. charge controlled capacitors, 7. current controlled inductors and 8. flux controlled inductors. Except for the sources, we further suppose the nonlinear elements to be time-invariant (although this is not a real constraint).

If we collect same types of elements into a single group, the b branch relations of a general circuit $\mathcal{C}$ composed of 2-terminal elements, are specified by

$$\begin{array}{lll} 1.\, u = e & 2.\, i = j & 3.\, u = \tilde{u}(i) \\ 4.\, i = \tilde{i}(u) & 5.\, q = \tilde{q}(u) & 6.\, u = \tilde{u}(q) \\ 7.\, \phi = \tilde{\phi}(i) & 8.\, i = \tilde{i}(\phi)\,. & \end{array} \qquad (3)$$

Here, e and j denote the source strengths, while a superscript refers to a nonlinear function description of a constitutive variable. Furthermore, q and ϕ are the electric charge and the magnetic flux, respectively. They are related to i and u by the auxiliary equations

$$q = \textstyle\int^t i d\tau \text{ and } \phi = \int^t u d\tau\,, \qquad (4)$$

where τ denotes a dummy variable. In general, all variables are a function of time t, explicitely denoted as $x = x(t)$ for any constitutive variable x.

The collection of the $(n-1)+b$ independent Kirchhoff-equations (2) and the b branch relations (3) constitute the $(n-1)+2b$ independent tableau-equations. Note that the number of equations equals the number of unknowns: $2b$ branch variables (u's and i's) plus $(n-1)$ note-to-datum voltages; the tableau formalism generates a well-posed problem.

We now assume that a dynamic nonlinear solution of the tableau-equations (2) and (3) is known. Next, we consider the same nonlinear circuit $\mathcal{C}$, but with small departures from the known solution, for example caused by small source variations. Then, the value of any variable $x(t)$ in the tableau-equations has to be replaced by a new value $x_{\mathcal{C}}(t) + \hat{x}(t)$, in which $x_{\mathcal{C}}(t)$ and $\hat{x}(t)$ denotes the known solution and a small variation, respectively.

As a consequence, the Kirchhoff-equations of $\mathcal{C}$ now become

$$\mathbf{A}(\boldsymbol{i}_{\mathcal{C}} + \hat{\boldsymbol{i}}) = \boldsymbol{0} \quad (KCL) \qquad (5a)$$

and

$$(\boldsymbol{u}_{\mathcal{C}} + \hat{\boldsymbol{u}}) - \mathbf{A}^T(\boldsymbol{v}_{\mathcal{C}} + \hat{\boldsymbol{v}}) = \boldsymbol{0} \quad (KVL)\,, \qquad (5b)$$

while the branch relations read

$$\begin{array}{ll} 1. u_{\mathcal{C}} + \hat{u} = e + \hat{e} & 5. q_{\mathcal{C}} + \hat{q} = \tilde{q}(u_{\mathcal{C}} + \hat{u}) \\ 2. i_{\mathcal{C}} + \hat{i} = j + \hat{j} & 6. u_{\mathcal{C}} + \hat{u} = \tilde{u}(q_{\mathcal{C}} + \hat{q}) \\ 3. u_{\mathcal{C}} + \hat{u} = \tilde{u}(i_{\mathcal{C}} + \hat{i}) & 7. \phi_{\mathcal{C}} + \hat{\phi} = \tilde{\phi}(i_{\mathcal{C}} + \hat{i}) \\ 4. i_{\mathcal{C}} + \hat{i} = \tilde{i}(u_{\mathcal{C}} + \hat{u}) & 8. i_{\mathcal{C}} + \hat{i} = \tilde{i}(\phi_{\mathcal{C}} + \hat{\phi}) \end{array} \qquad (6)$$

with auxiliary equations

$$q_{\mathcal{C}} + \hat{q} = \textstyle\int^t (i_{\mathcal{C}} + \hat{i}) d\tau \text{and} \phi_{\mathcal{C}} + \hat{\phi} = \int^t (u_{\mathcal{C}} + \hat{u}) d\tau. \quad (7)$$

Since the variations are supposed to be small, we may neglect higher order terms in the Taylor-expansion for any nonlinear function $y = \tilde{y}(x)$ of a constitutive variable x. Thus

$$y_{\mathcal{C}} + \hat{y} = \tilde{y}(x_{\mathcal{C}} + \hat{x}) = \tilde{y}(x_{\mathcal{C}}) + (d\tilde{y}\,/\,dx)_{\mathcal{C}}\hat{x}\,, \quad (8)$$

in which the derivative $(d\tilde{y}\,/\,dx)$ is evaluated along the known source-dependent solution $x_{\mathcal{C}} = x_{\mathcal{C}}(t)$.

Next, we substract the tableau-equations (2) and (3) line-by-line from the tableau-equations (5) and (6), respectively. Then, in view of the approximation (8), we arrive at the following tableau-equations for the small variations

$$\mathbf{A}\hat{\boldsymbol{i}} = \boldsymbol{0}\ \ (KCL) \text{ and } \hat{\boldsymbol{u}} - \mathbf{A}^T\hat{\boldsymbol{v}} = \boldsymbol{0}\ \ (KVL)\,, \quad (9)$$

and

$$\begin{array}{lll} 1.\, \hat{u} = \hat{e} & 2.\, \hat{i} = \hat{j} & 3.\, \hat{u} = r\hat{i} \\ 4.\, \hat{i} = g\hat{u} & 5.\, \hat{q} = c\hat{u} & 6.\, \hat{u} = s\hat{q} \\ 7.\, \hat{\phi} = l\hat{i} & 8.\, \hat{i} = \gamma\hat{\phi} & \end{array} \tag{10}$$

with auxiliary equations

$$\hat{q} = \textstyle\int^t \hat{i} d\tau \text{ and } \hat{\phi} = \int^t \hat{u} d\tau \, . \tag{11}$$

In (10), the incremental quantities r, g, c, l and γ are called the differential resistance $[\Omega]$, - conductance $[S]$, - capicitance $[F]$, - elastance $[F^{-1}]$, - inductance $[H]$ and - inverse inductance $[H^{-1}]$, respectively. They are given by

$$\begin{array}{ll} r(t) = (d\tilde{u} \,/\, di)_{\mathcal{C}} & s(t) = (d\tilde{u} \,/\, dq)_{\mathcal{C}} \\ g(t) = (d\tilde{i} \,/\, du)_{\mathcal{C}} & l(t) = (d\tilde{\phi} \,/\, di)_{\mathcal{C}} \\ c(t) = (d\tilde{q} \,/\, du)_{\mathcal{C}} & \gamma(t) = (d\tilde{i} \,/\, d\phi)_{\mathcal{C}} \end{array} \tag{12}$$

where the notation underlines that the derivatives are all evaluated along the known source-dependent solution $x_{\mathcal{C}} = x_{\mathcal{C}}(t)$ of the orginal nonlinear circuit $\mathcal{C}$ and hence are time-dependent. They are subsequently interpreted as the constitutive coefficients of linear time-varying circuit elements.
Finally, in view of the tableau-equations (2) and (3), it is clearly observed that the tableau-equations (9) and (10) define a new, LTV-variational circuit $\hat{\mathcal{C}}$ characterized by 1. the same topology as $\mathcal{C}$ (same circuit structure matrix $\mathbf{A}$) and 2. any element of $\mathcal{C}$ is replaced by an associated LTV circuit element in $\hat{\mathcal{C}}$.
Before generalizing this result to circuits with nonlinear three- and more- terminal elements, we first present a simple example.
In the nonlinear first order circuit $\mathcal{C}$ of figure 1.a, we want to study the influence of a small-signal source $\hat{e}$ upon the nonlinear behavior. The terminal relations of the constituent elements are given by

$$\begin{array}{l} R_1 : i = G_1 u \;\; R_2 : i = \tilde{i}_2(u) \quad R_3 : i = \tilde{i}_3(u) \\ C : u = \tilde{u}(q) \text{ with } q = \int^t i d\tau \, . \end{array} \tag{13}$$

Given a solution of $\mathcal{C}$ with $\hat{e} = 0$, the associated LTV small-signal circuit $\hat{\mathcal{C}}$ is known, too. Due to its linearity, the topology of $\hat{\mathcal{C}}$ can be drastically reduced by using Norton's theorem together with the rule for parallel connections of linear conductances. The result is shown in figure 1.b.
The derivation of the LTV variational circuit pertaining to circuits containing nonlinear N-terminal elements is essentially the same as before. Taking one terminal as datum node, a N-terminal element defines $(N-1)$ branches. Each corresponding branch relation interacts with the other ones. For example, a nonlinear 3-terminal resistor generates two coupled branch relations. For them, we have six possible representations (a choice of two out of four variables). If we take the voltage controlled representation (e.g. a resistive Ebers-Moll model for a common base bipolar transistor), the two branch relations

$$i_1 = \tilde{i}_1(u_1, u_2) \quad \text{and} \quad i_2 = \tilde{i}_2(u_1, u_2) \tag{14}$$

give rise to the following LTV small-signal branch relations

$$\hat{i}_1 = g_{11}\hat{u}_1 + g_{12}\hat{u}_2 \quad \text{and} \quad \hat{i}_2 = g_{21}\hat{u}_1 + g_{22}\hat{u}_2, \tag{15}$$

in which the incremental quantities

$$g_{ij}(t) = (\partial\tilde{i}_i \,/\, \partial u_j)_{\mathcal{C}} \qquad (i, j = 1, 2) \tag{16}$$

denote differential conductances, evaluated along the source-dependent solution of the original nonlinear circuit $\mathcal{C}$. Hence, they define the constitutive coefficients of a LTV voltage controlled 3-terminal resistor.

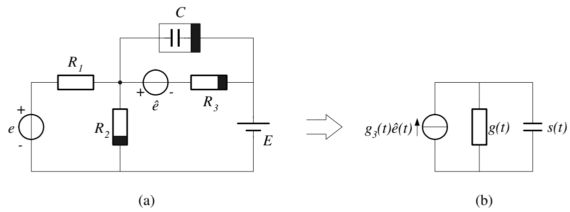

Fig. 1. Original nonlinear circuit $\mathcal{C}$ (a) and reduced LTV small-signal circuit, with $g = G_1 + g_2 + g_3$ (b).

III. Time-discrete LTV models

In this section it is shown that the LTV small-signal circuit is easily converted into a time-discrete resistive circuit. To that aim, the LTV resistive elements are taken at their value at discrete times t_k, while the dynamic LTV elements are replaced by discrete recursive resistive elements.
To start with, consider a LTV capacitor with voltage controlled branch relation (10.5). Combined with the auxiliary equation (11) yields the $u - i$-relation

$$c(t)u(t) = \textstyle\int^t i(\tau) d\tau \, , \tag{17}$$

where we have suppressed the notation for small variations. For two successive times t_k and t_{k+1} it follows

$$c_{k+1}u_{k+1} = c_k u_k + \int_{t_k}^{t_{k+1}} i(\tau) d\tau \, , \tag{18}$$

in which index k refers to time t_k. Next, the integral in (18) is approximated by a suitable numerical integration rule. For example, the backward Euler-integration rule with step size h yields

$$c_{k+1}u_{k+1} = c_k u_k + h i_{k+1} \; , \tag{19}$$

which is subsequently rewritten as

$$i_{k+1} = (h^{-1}c_{k+1})u_{k+1} - (h^{-1}c_k)u_k \; . \tag{20}$$

Finally, this expression is recognized as the $u - i$-relation of a recursive (dynamic) resistive one-port at time t_{k+1}, as shown in figure 2.a. In analogy, for a

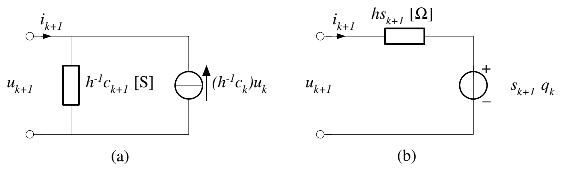

Fig. 2. Time-discrete LTV capacitor (Backward Euler). Voltage controlled (a) and charge controlled (b).

charge controlled LTV capacitor with branch relation (10.6) the recursive resistive port of figure 2.b is obtained. By setting $c_k = s_k^{-1}$, it is easily seen that the one-ports in figure 2 are equivalent. For LTV inductors, dual time-discrete models are obtained.
Finally, by replacing each LTV element in a LTV small-signal circuit by an appropriate time-discrete element, there results a time-discrete resistive circuit that can be solved recursively.

IV. Conclusions

For a given nonlinear electronic circuit, an associated linear time-varying small-signal circuit is formally derived by the tableau method. Due to its linearity, linear circuit theory may be applied. It opens the possibility of simplifying the circuit topology. Alternatively, the LTV solution can be found recursively by replacing each LTV circuit element by an appropriate time-discrete resistive model.

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