ITERATION SCHEMES FOR THE MODAL SOLUTIONS OF LINEAR TIME-VARYING SYSTEMS

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ABSTRACT

On the basis of a mode-vector representation, we show that its time-varying amplitudes and frequencies can respectively be obtained by diagonalizing the time-varying system equations. Next, we reformulate an explicit iteration scheme that was earlier proposed by Wu. Then, the missing convergence proof is given. Moreover, we present a new and implicit algorithm that is closely related to that of Wu. In both algorithms, the time-varying system matrix is gradually diagonalized by successive algebraic similarity transformations.

1. INTRODUCTION

Apart from their own significance, linear time-varying systems can also result from modelling all kinds of physical systems. For example, when studying the local behaviour of nonlinear electronic circuits, the Jacobian matrix along a bias trajectory constitutes the time-varying system matrix [1]. In principle, the solutions of the describing linear system of differential equations can be obtained by diagonalizing the system equations [2]. However, a direct implementation of such an approach necessarily involves the solution of a Riccati differential matrix equation [3]. In [4], this is worked out in detail for two-dimensional systems.

An alternative for this procedure was earlier proposed in a paper by Wu [5]. There, the author introduced an iteration scheme that diagonalizes the system equations without the need for solving Riccati-equations. However, Wu's formulation of the algorithm seems quite cumbersome, mainly because a convergence proof was missing. Perhaps it could therefore happen that in a next paper the algorithm was qualified as a trial-and-error method without any further comment [6].

In this paper, we present a complete proof of Wu's algorithm. The conditions which have to be satisfied for convergence are also given. Moreover, we present a new algorithm that is closely related to that of Wu. In both algorithms, the time-varying system matrix is gradually diagonalized by successive algebraic similarity transformations. Hence, in the proces of computation the matrices involved are assumed to depend parametrically on the time.

Following Wu [5], in the next section the problem is formulated with a mode-vector representation for the solutions as a starting point. Then, a subsequent substitution in the system equations results into what we call the dynamic eigenvalue problem. It is, indeed, a generalization of the classical, algebraic eigenvalue problem for systems with constant elements. Next, we show that the modal solutions can be obtained by a coordinate transformation that diagonalizes the system equations.

In Section 3, the iteration scheme earlier proposed by Wu is reformulated and subsequently identified as an explicit algorithm. This is followed by a convergence proof. It turns out that the uniform convergence of the rate of change of the successive algebraic similarity transformations is crucial. As a variant of Wu's algorithm, in Section 4 an implicit diagonalization algorithm is derived. It is proved that the conditions for convergence are the same as for the explicit scheme.

2. STATEMENT OF THE PROBLEM

Consider the homogeneous linear system of time-varying differential equations

$$\dot{\boldsymbol{x}} = \mathbf{A}(t)\,\boldsymbol{x} \quad , \tag{1}$$

where $\mathbf{A}$ denotes a given $n \times n$ system matrix whose elements are continuously differentiable functions of the time t, while $\boldsymbol{x}$ is an unknown time-dependent n-vector. Following Wu [5], we look for n independent solutions of the modal form

$$\boldsymbol{x}(t) \;=\; \boldsymbol{u}(t)\,\exp(\gamma(t)) \quad . \tag{2}$$

Here, $\boldsymbol{u}(t)$ can be interpreted as a time-dependent amplitude vector, while the scalar funtion $\gamma(t)$ can be associated with a time-dependent frequency $\lambda(t)$, given by

$$\lambda(t) \;=\; \dot{\gamma}(t) \Longleftrightarrow \gamma(t)=\gamma(t_0)\;+\;\int_{t_0}^{t}\lambda(\tau)\,\mathrm{d}\tau \tag{3}$$

where t_0 is the initial time. Substitution of (2) into (1) results into the following expression for the unknowns $\boldsymbol{u}(t)$ and $\lambda(t)$ (compare [5], [6] and [7])

$$[\mathbf{A}(t)-\lambda(t)\mathbf{I}]\boldsymbol{u}(t) \;=\; \dot{\boldsymbol{u}}(t) \quad , \tag{4}$$

in which $\mathbf{I}$ denotes the identity. Equation (4) constitutes what we call the dynamic eigenvalue problem associated with (1). Then, the quantities $\lambda(t)$ and $\boldsymbol{u}(t)$ can be identified as a dynamic eigenvalue and a dynamic eigenvector, respectively. A pair $\{\lambda\,,\;\boldsymbol{u}\}$ such that (4) is satisfied, is called a dynamic eigenpair of (1). Note that there is a distinguished difference with the classical eigenvalue problem. There, both λ and $\boldsymbol{u}$ are constants, so the right-hand side of (4) becomes zero. In particular, it has to be understood that although an algebraic eigenpair of the time-varying matrix $\mathbf{A}(t)$ is time-dependent too, in general it doesn't have the direct physical interpretation as its dynamical counterpart.
Next, we replace $\boldsymbol{x}(t)$ by $\boldsymbol{y}(t)$ according to the coordinate transformation

$$\boldsymbol{x}(t) \;=\; \mathbf{L}(t)\,\boldsymbol{y}(t) \tag{5}$$

that results into the new linear system of time-varying equations

$$\dot{\boldsymbol{y}}(t) \;=\; \mathbf{B}(t)\,\boldsymbol{y}(t) \quad . \tag{6}$$

As a consequence, the system matrix $\mathbf{B}$ is given by [8]

$$\mathbf{B} \;=\; \mathbf{L}^{-1}\,\mathbf{A}\,\mathbf{L}\;-\;\mathbf{L}^{-1}\,\dot{\mathbf{L}} \quad . \tag{7}$$

By combining (2) and (5), it is observed that the n independent solutions $\boldsymbol{y}$ of (6) each have the modal form (2) with time-varying amplitudes $(\mathbf{L}^{-1}\,\boldsymbol{u})$, while the time-varying frequencies λ of $\boldsymbol{x}$ are shared by $\boldsymbol{y}$. This observation is reflected in the transformed dynamic eigenvalue problem, obtained from (6) after a substitution of (2) in (5) as

$$(\mathbf{B}-\lambda\mathbf{I})(\mathbf{L}^{-1}\boldsymbol{u}) \;=\; (\mathbf{L}^{-1}\boldsymbol{u})^{\cdot} \quad , \tag{8}$$

by which it follows that $\{\lambda\,,(\mathbf{L}^{-1}\boldsymbol{u})\}$ is a dynamic eigenpair of (6). Hence, the transformation $\mathbf{L}$ preserves the dynamic eigenvalues of the original system (1). For that reason we call $\mathbf{L}$, in analogy with the algebraic eigenvalue problem a dynamic similarity transformation, while the matrices $\mathbf{A}$ and $\mathbf{B}$ are called dynamic similar.
As in the algebraic case, the problem of solving (1) is drastically simplified if $\mathbf{L}$ can be chosen such that the system matrix $\mathbf{B}(t)$ equals a diagonal matrix $\Lambda(t)$. Then, system (6) becomes uncoupled and as a consequence, the time-varying amplitudes $(\mathbf{L}^{-1}\,\boldsymbol{u})$ of the solutions $\boldsymbol{y}$ can be chosen as constants. Thus, in that particular situation the right-hand side of (8) becomes zero. Now, it is immediately clear that the dynamic eigenvalues $\lambda(t)$ then will appear on the main diagonal of $\Lambda(t)$, as is always the case in the algebraic analogon.
In the next two sections, we respectively deal with two closely related iteration schemes that each start with the known system matrix $\mathbf{A}$, ending up with the requested matrices Λ and $\mathbf{L}$. The basic idea of the first algorithm was originally thrown up by Wu [5]. However, until now a complete proof was missing.

3. EXPLICIT DIAGONALIZATION

Let $\mathbf{A}(t)$ be dynamic similar with the diagonal matrix $\Lambda(t)$. Then, there exist a dynamic similarity transformation $\mathbf{L}(t)$ such that

$$\Lambda \;=\; \mathbf{L}^{-1}\,\mathbf{A}\,\mathbf{L}\;-\;\mathbf{L}^{-1}\,\dot{\mathbf{L}} \quad . \tag{9}$$

Note that the colums of $\mathbf{L}$ are just the dynamic eigenvectors of (1). Next, this expression is alternatively written as

$$\Lambda \;=\; \mathbf{L}^{-1}\,(\mathbf{A}\;-\;\dot{\mathbf{L}}\,\mathbf{L}^{-1}\,)\mathbf{L} \quad , \tag{10}$$

which can subsequently be interpreted as an algebraic similarity transformation of the matrix $(\mathbf{A}\;-\;\dot{\mathbf{L}}\,\mathbf{L}^{-1}\,)$. This suggests the following iteration scheme (compare [5])

$$\begin{aligned} \Lambda_j \quad &= \quad \mathbf{S}_j^{-1}\,(\mathbf{A}\;-\;\dot{\mathbf{S}}_{j-1}\,\mathbf{S}_{j-1}^{-1})\mathbf{S}_j \qquad (j=1,2,\ldots) \\ &\quad\;\, \text{with} \qquad \mathbf{S}_{\mathrm{o}}\;=\;\mathbf{I} \quad , \end{aligned} \tag{11}$$

where $\mathbf{S}_j(t)$ is such that $\Lambda_j(t)$ is diagonal. Hence, $\mathbf{S}_j(t)$ constitutes a classical similarity transformation of the matrix $(\mathbf{A}\;-\;\dot{\mathbf{S}}_{j-1}\,\mathbf{S}_{j-1}^{-1})$ that now depends parametrically on the time. Thus, its time-parameterized algebraic eigenvectors go into the colums of $\mathbf{S}_j$. Thereby, it is expected that $\mathbf{S}_j$ and Λ_j converge to $\mathbf{L}$ and Λ, respectively. Under certain conditions, it turns out that this conjecture is valid indeed. Note that (11) gives an explicit expression for the successive approximations Λ_j of Λ. Therefore, (11) is called an explicit iteration scheme.
We now present the main result of this paper.
Theorem
If, and only if the row $\{\dot{\mathbf{S}}_j\}_{j=1}^{\infty}$ is uniformly convergent, the algorithm defined by (11) converges. In particular, the diagonal matrix Λ and the transformation matrix $\mathbf{L}$ in (9) are respectively obtained as

$$\Lambda(t) \;=\; \lim_{j\to\infty}\Lambda_j(t) \quad , \tag{12}$$

$$\mathbf{L}(t) = \lim_{j\to\infty} \mathbf{S}_j(t) \quad . \tag{13}$$

In order to prove the theorem, we first derive the next identity.

Lemma 1

$$\Lambda = \mathbf{R}_j^{-1}\,\Lambda_j\,\mathbf{R}_j - \mathbf{R}_j^{-1}\,\dot{\mathbf{R}}_j + \mathbf{E}_j \quad , \tag{14}$$

where the matrices $\mathbf{R}_j(t)$ and $\mathbf{E}_j(t)$ are respectively given by

$$\mathbf{R}_j = \mathbf{S}_j^{-1}\,\mathbf{L} \quad , \tag{15}$$

$$\mathbf{E}_j = \mathbf{L}^{-1}\,(\dot{\mathbf{S}}_{j-1}\mathbf{S}_{j-1}^{-1} - \dot{\mathbf{S}}_j\mathbf{S}_j^{-1})\,\mathbf{L} \quad . \tag{16}$$

Thus, up to the error $\mathbf{E}_j$, the diagonal matrices Λ and Λ_j are dynamic similar with dynamic similarity transformation $\mathbf{R}_j$.

Proof

First, $\mathbf{A}$ is eliminated from (11) and (10). This yields

$$\Lambda = \mathbf{L}^{-1}\,(\mathbf{S}_j\Lambda_j\mathbf{S}_j^{-1} + \dot{\mathbf{S}}_{j-1}\mathbf{S}_{j-1}^{-1} - \dot{\mathbf{L}}\mathbf{L}^{-1})\,\mathbf{L} \quad , \tag{17}$$

or, after rearranging

$$\begin{aligned}\Lambda \;=\; & (\mathbf{L}^{-1}\mathbf{S}_j)[\Lambda_j - \mathbf{S}_j^{-1}\dot{\mathbf{L}}\mathbf{L}^{-1}\mathbf{S}_j + \\ & + \mathbf{S}_j^{-1}\dot{\mathbf{S}}_{j-1}\mathbf{S}_{j-1}^{-1}\mathbf{S}_j](\mathbf{S}_j^{-1}\mathbf{L}) \quad .\end{aligned} \tag{18}$$

Next, the identity

$$(\mathbf{S}_j^{-1}\mathbf{L})^{\cdot}\,(\mathbf{S}_j^{-1}\mathbf{L})^{-1} = \mathbf{S}_j^{-1}\dot{\mathbf{L}}\mathbf{L}^{-1}\mathbf{S}_j - \mathbf{S}_j^{-1}\dot{\mathbf{S}}_j \tag{19}$$

is substituted into (18), resulting into the expression

$$\begin{aligned}\Lambda \;=\; & (\mathbf{L}^{-1}\mathbf{S}_j)[\Lambda_j - (\mathbf{S}_j^{-1}\mathbf{L})^{\cdot}\,(\mathbf{S}_j^{-1}\mathbf{L})^{-1}](\mathbf{S}_j^{-1}\mathbf{L}) + \\ & (\mathbf{L}^{-1}\mathbf{S}_j)[\mathbf{S}_j^{-1}\dot{\mathbf{S}}_{j-1}\mathbf{S}_{j-1}^{-1}\mathbf{S}_j - \mathbf{S}_j^{-1}\dot{\mathbf{S}}_j](\mathbf{S}_j^{-1}\mathbf{L})\,,\end{aligned} \tag{20}$$

in which the second term in the right-hand side is recognized as the errror matrix $\mathbf{E}_j$. Now, the lemma follows immediately. □

Proof of the Theorem

Necessity: Since $\{\dot{\mathbf{S}}_j\}_{j=1}^{\infty}$ is uniformly convergent, it follows by integration that $\mathbf{S}_j$ has a nonzero limit for $j \to \infty$. So, $\lim_{j\to\infty} \mathbf{S}_j^{-1}$ exists. As a consequence, it follows from (16) $\lim_{j\to\infty} \mathbf{E}_j = \mathbf{0}$. Hence, the algorithm converges.

Next, notice that $\lim_{j\to\infty} \mathbf{S}_j^{-1} = (\lim_{j\to\infty} \mathbf{S}_j)^{-1}$, while with respect to the assumed uniform convergence we also have $\lim_{j\to\infty} \dot{\mathbf{S}}_j = (\lim_{j\to\infty} \mathbf{S}_j)^{\cdot}$. Thus, we obtain from (14) for $j \to \infty$

$$\Lambda = \mathbf{R}^{-1}(\lim_{j\to\infty} \Lambda_j)\mathbf{R} - \mathbf{R}^{-1}\dot{\mathbf{R}} \quad , \tag{21}$$

where $\mathbf{R} = \lim_{j\to\infty} \mathbf{R}_j$. Now, observe from (21) that Λ and $(\lim_{j\to\infty} \Lambda_j)$ are dynamic similar with $\mathbf{R}$ as dynamic similarity transformation matrix. As a consequence, Λ and $(\lim_{j\to\infty} \Lambda_j)$ share the same (dynamic) eigenvalues. And since both matrices are diagonal, we readily conclude to the first statement (12) of the Theorem. Next, it follows from (21) that $\mathbf{R}$ has to satisfy

$$\dot{\mathbf{R}} = \Lambda\mathbf{R} - \mathbf{R}\Lambda \tag{22}$$

from which the elements $r_{ij}(t)$ of $\mathbf{R}$ follows as $r_{ij} = c_{ij}\exp(\gamma_i - \gamma_j)$ with c_{ij} arbitarily constants, some of them may be zero. So, $\mathbf{L}$ is obtained from (15) for $j \to \infty$ as

$$\mathbf{L}(t) = \{\lim_{j\to\infty} \mathbf{S}_j(t)\}\mathbf{R}(t) \quad . \tag{23}$$

Finally, it can easily be shown that the fundamental matrices corresponding to the two different expressions (13) and (23) for $\mathbf{L}$ are identical. Therefore, we can chose $\mathbf{R} = \mathbf{I}$ without loss of generality. Now, the second statement (13) of the Theorem is also proved.

Sufficiently: Since the algorithm converges, we have from (14) $\lim_{j\to\infty} \mathbf{E}_j = \mathbf{0}$. Furthermore, we conclude with (12) and (23) with $\mathbf{R}$ satisfying (22) $\dot{\mathbf{R}} = \lim_{j\to\infty} \dot{\mathbf{R}}_j$. Since $\mathbf{R} = \lim_{j\to\infty} \mathbf{R}_j$, it follows that the row $\{\mathbf{R}_j\}_{j=1}^{\infty}$ is uniformly convergent. Then, on account of (15), the row $\{(\mathbf{S}_j^{-1})^{\cdot}\}_{j=1}^{\infty}$ is uniformly convergent, too. Finally, by using the identity $(\mathbf{S}_j^{-1})^{\cdot} = -\mathbf{S}_j^{-1}\dot{\mathbf{S}}_j\mathbf{S}_j^{-1}$ for $j \to \infty$, we conclude that $\{\dot{\mathbf{S}}_j\}_{j=1}^{\infty}$ is uniformly convergent. □

4. IMPLICIT DIAGONALIZATION

In order to derive an implicit iteration scheme, the system matrix $\mathbf{A}(t)$ is first parametrically diagonalized by an algebraic similarity transformation $\mathbf{Q}_1(t)$. Thus

$$\bar{\Lambda}_1 = \mathbf{Q}_1^{-1}\mathbf{A}\mathbf{Q}_1 \quad , \tag{24}$$

where $\bar{\Lambda}_1(t)$ is diagonal. Substitution of (24) in (10) results into

$$\Lambda = \mathbf{L}^{-1}(\mathbf{Q}_1\bar{\Lambda}_1\mathbf{Q}_1^{-1} - \dot{\mathbf{L}}\mathbf{L}^{-1})\mathbf{L} \tag{25}$$

which in turn is rewritten as

$$\begin{aligned}\Lambda \;=\; & (\mathbf{Q}_1^{-1}\mathbf{L})^{-1}(\bar{\Lambda}_1 - \mathbf{Q}_1^{-1}\dot{\mathbf{Q}}_1)(\mathbf{Q}_1^{-1}\mathbf{L}) - \\ & (\mathbf{Q}_1^{-1}\mathbf{L})^{-1}(\mathbf{Q}_1^{-1}\mathbf{L})^{\cdot} \quad ,\end{aligned} \tag{26}$$

where we again have used idendity (19) with $\mathbf{Q}_1$ instead of $\mathbf{S}_j$. Now, it is observed from (26) that the matrix $(\bar{\Lambda}_1 - \mathbf{Q}_1^{-1}\dot{\mathbf{Q}}_1)$ is dynamic similar with Λ. As a next step, we repeat the procedure by parametrically diagonalizing the matrix $(\bar{\Lambda}_1 - \mathbf{Q}_1^{-1}\dot{\mathbf{Q}}_1)$ by an algebraic similarity transformation $\mathbf{Q}_2(t)$, and so on. This brings us to the following implicit iteration scheme

$$\bar{\Lambda}_j = \mathbf{Q}_j^{-1}(\bar{\Lambda}_{j-1} - \mathbf{Q}_{j-1}^{-1}\dot{\mathbf{Q}}_{j-1})\mathbf{Q}_j \qquad (j = 1, 2, \ldots) \tag{27}$$

with

$$\bar{\Lambda}_0 = \mathbf{A} \qquad \text{and} \qquad \mathbf{Q}_0 = \mathbf{I} \quad , \tag{28}$$

where the diagonal matrices $\bar{\Lambda}_j(t) \quad (j = 1, 2, \cdots)$ are up to an error $\mathbf{Q}_j^{-1}\dot{\mathbf{Q}}_j$ dynamic similar with $\Lambda(t)$.
By executing the algorithm, it is readily seen that the successive algebraic similarity transformation matrices $\mathbf{Q}_j$ are multiplied by each other, resulting into a transformation matrix $\mathbf{P}_j(t)$, defined by

$$\mathbf{P}_j = \mathbf{Q}_1\mathbf{Q}_2 \ldots \mathbf{Q}_j \qquad (j = 1, 2 \ldots) \quad . \tag{29}$$

As in the preceding section, it is now expected that under conditions $\mathbf{P}_j$ converges for $j \to \infty$ to $\mathbf{L}$ and $\bar{\Lambda}_j$ to Λ, respectively.
The proof of this conjecture is easely given, if one is aware of the relation between the implicit and explicit algorithm as expressed by the following lemma.
Lemma 2
The relation between the explicit and the implicit diagonalization algorithm as defined by (11) and (27) respectively, is given by

$$\mathbf{S}_j = \mathbf{P}_j \qquad \text{and} \qquad \Lambda_j = \bar{\Lambda}_j \tag{30}$$

where $\mathbf{P}_j$ is given by (29).
Proof
First, by comparing (27) with (11) for $j = 1$, it follows that $\bar{\Lambda}_1 = \Lambda_1$ and $\mathbf{S}_1 = \mathbf{Q}_1$. This result is subsequently substituted in (27) for $j = 2$, followed by a substitution of (11) for $j = 1$. Then, we easily obtain $\bar{\Lambda}_2 = \Lambda_2$ and $\mathbf{S}_2 = \mathbf{Q}_1\mathbf{Q}_2$. Next, we repeat the procedure for $j = 3$, yielding after straightforward calculation $\bar{\Lambda}_3 = \Lambda_3$ and $\mathbf{S}_3 = \mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3$, and so on. Finally, by mathematical induction (30) follows. □
By using Lemma 2 in combination with the Theorem, we have proved the conjecture.

5. CONCLUSIONS

It is shown that the iteration schemes perform successive coordinate transformations that gradually result into that coordinate system in which an observer travels along the trajectory of the final diagonalized system. Only then, the associated modes have constant amplitudes while they are completely uncoupled. Of course, it remains to be investigated for which class of systems the row $\{\dot{\mathbf{S}}_j\}_{j=1}^{\infty}$ is uniformly convergent indeed.
The following observation suggests such a class of systems. Systems for which $\dot{\mathbf{S}}_j \to \mathbf{0}$, as a consequence have the property that the successive algebraic similarity matrices $\mathbf{S}_j$ approach a constant matrix. Thus, for that particular class of systems, the time-parameterized algebraic eigenvectors approach constant vectors, too. This leads to the indication that the algorithms converge if in the proces of diagonalization the updated system matrix $(\mathbf{A} - \dot{\mathbf{S}}_j\,\mathbf{S}_j^{-1})$ in (11) gradually shows a slowlier varying behaviour [9].
As a final remark, we note that following the method of this paper, the dynamic eigenpairs can be determined in an unique way. However, in the approach outlined in an earlier paper, we pointed out that there is a dependence on the initial value of an associated second-order Riccati differential equation [4]. Such a Riccati equation has one stable and one unstable equilibrium point. By choosing the initial condition in a stable region, both methods yield identical results.

6. REFERENCES

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