BEHAVIOUR OF DYNAMIC EIGENPAIRS IN SLOWLY-VARYING SYSTEMS

P. van der Kloet and F.L. Neerhoff

Delft University of Technology
Faculty of Information Technology and Systems
Laboratory of Circuits and Systems
P.O.Box 5031, 2600 DA Delft, The Netherlands
Phone: +31 15 2781372
Fax: +31 15 2786190
E-mail: F.L.Neerhoff@its.tudelft.nl

ABSTRACT

Slowly-varying systems can be approximated by time-invariant systems. Such an approximated slow-varying system can be described with algebraic eigenpairs. These eigenpairs determine the (form and) behaviour of the fundamental matrix. On its turn, this fundamental matrix is an approximation of the fundamental matrix of the original time-varying system. However, the properties of the latter one are described by so called dynamic eigenpairs. In this paper, it is shown how the dynamic eigenpairs are approximated by the classical algebraic eigenpairs.

1. INTRODUCTION

One of the most remarkable results in the theory of linear differential systems with slowly-varying coefficients is given by Desoer [1] in a one-page article. The so called "frozen time" behaviour appears in variational equations with respect to the underlying nonlinear dynamical systems when this original system has a relatively slowly varying bias signal [2]. Applications of the "frozen time" approach in the field of stabilizing feedback systems are given among others by Kamen et.al. [3]. Even in recent textbooks on linear systems [4] and linear differential equations [5], attention has been given to this basic result on the frozen time approach.

Recently, Solo [6] found that the stable solutions of slowly time-varying linear systems are allowed to have time-dependent eigenvalues in the right halfplane as long as on the average they are strictly in the left halfplane. In [7] an example has been given of a system which is not slowly varying but which also shows this behaviour of an eigenvalue which is allowed to "wander" in the right halfplane as long as its mean value is located in the left halfplane. So the conclusions of Solo can be broadened.

From another point of view, Kamen [8] deduced that the two eigenvalues of a second order dynamical system with time-varying coefficients do not behave like complex conjugated numbers. This indicates that the role and use of algebraic eigenvalues has to be questionized.

A same type of problem appears in the asymptotic theory for ordinary differential equations [9]. There, also, a suggestion for the improvement of the algebraic eigenvalues is discussed. Nevertheless, a physical base for these improvements is not given.

A succesful attempt to give such an argument is given by Wu [10, 11]. In his papers, he introduces the concepts of non-parametric arbitrarily time-dependent principal modes, eigenvalues and eigenvectors. To distinguish them from the classical algebraic, parametric eigenvalues, we will use the prefix dynamic (see also [12]). Moreover, Wu formulates the dynamic eigenvalue problem, which will yield the dynamic eigenvalues and dynamic eigenvectors.

The dynamic principal modes appear by assuming that the linear time-varying dynamical system has solutions in the form of single exponential functions. Each exponent itself is in general a nonlinear function of time. For a linear time-invariant system this nonlinear function reduces to a linear function with the eigenvalue as slope. This opens a perspective to generalize the concept of the classical algebraic eigenvalues to dynamic eigenvalues; they are simply defined as the time-derivative of the nonlinear exponent in the dynamic mode. (see also [12]).

An example of a two-dimensional system is presented in [13]. This example explains the problems which are reported in [8]. In [14, 15] it is reported how the dynamic eigenvalues generalize the Floquet-numbers. In [12], an improvement on a result of Wu [11] is presented. This result indicates in some sense that the classical algebraic eigenvalue can be considered, under convergence conditions, as a limiting value of iterated dynamic eigenvalues.

In [16] an alternative for the classical characteristic equation will be derived. This requires a similarity transformation applied on the systemmatrix. The similarity transformation is performed using a shear transformationmatrix.

In this paper, a second extension to linear time-varying

dynamical systems is discussed. This approach generalizes the results in [8] on time-continuous dynamical systems. The extension, to be presented, also justifies the use of algebraic eigenvalues in slowly-varying systems in stead of dynamic eigenvalues. Further, it explains the use of algebraic eigenvalues when the systemmatrix approaches a constant matrix for time going to infinity. Moreover, it seems to explain the requirements for convergence in the iteration procedure given in [12].
In this paper a generalized characteristic equation for second order systems will be discussed. It is shown that for slowly-varying systems the classical theory motivates an approximation of the theory on dynamic eigenvalues.

2. MATHEMATICAL BACKGROUND

Consider a two-dimensional linear time-varying differential system with

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad . \tag{1}$$

Assume that

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \boldsymbol{u}_1 \exp(\gamma_1(t)) \tag{2}$$

satisfies (1). Substitution of (2) into (1) yields the dynamic eigenvalue problem [5, 11, 12]

$$\begin{bmatrix} a_{11}-\lambda_1 & a_{12} \\ a_{21} & a_{22}-\lambda_1 \end{bmatrix} \boldsymbol{u}_1 = \dot{\boldsymbol{u}}_1 \quad , \tag{3}$$

where

$$\lambda_1(t) = \dot{\gamma}_1(t) \quad \Leftrightarrow \quad \gamma_1(t) = \int_0^t \lambda_1(\tau)\mathrm{d}\tau \tag{4}$$

with

$$\gamma_1(0) = 0 \quad . \tag{5}$$

A pair $(\lambda_1, \boldsymbol{u}_1)$ such that (3) is satisfied, is called a dynamic eigenpair with λ_1 a dynamic eigenvalue and $\boldsymbol{u}_1$ a dynamic eigenvector, respectively [12]. Assume further

$$\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ l(t) \end{bmatrix} \tag{6}$$

satisfies (3). Then we must have

$$\begin{array}{rcrcl} (a_{11}-\lambda_1) & + & a_{12}l & = & 0 \\ a_{21} & + & (a_{22}-\lambda_1)l & = & \dot{l} \end{array} \quad . \tag{7}$$

Elimination of l from (7) yields

$$a_{12}a_{21} - (a_{11}-\lambda_1)(a_{22}-\lambda_1) = a_{12}\dot{l} \quad . \tag{8}$$

This is a generalization of the characteristic equation for time invariant systems. There, of course, we have $\dot{l} = 0$. We can also eliminate λ_1 from (7). Then

$$\dot{l} = -a_{12}l^2 - (a_{11}-a_{22})l + a_{21} \quad . \tag{9}$$

This is a Riccati differential equation. It is clearly equivalent to the characteristic equation for time invariant systems.
If l satisfies (9), then the state transformation

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ l & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{10}$$

leads (1) to

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & a_{12} \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad , \tag{11}$$

where

$$\lambda_2(t) = -la_{12} + a_{22} \quad . \tag{12}$$

With λ_2 instead of λ_1 we can write (7) as

$$\begin{array}{rcrcl} (\lambda_2-a_{22}) & + & a_{12}l & = & 0 \\ a_{21} & + & (\lambda_2-a_{11})l & = & \dot{l} \end{array} \quad . \tag{13}$$

Or, compactly, as [16]

$$\begin{bmatrix} l & -1 \end{bmatrix} \begin{bmatrix} \lambda_2-a_{11} & -a_{12} \\ -a_{21} & \lambda_2-a_{22} \end{bmatrix} = \begin{bmatrix} \dot{l} & -1 \end{bmatrix} . \tag{14}$$

Finally, remark that the state transformation

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \tag{15}$$

with $u(t)$ the solution of

$$\dot{u} = (\lambda_1-\lambda_2)u + a_{12} \quad , \tag{16}$$

will give for (11)

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \quad . \tag{17}$$

By introduction of

$$\gamma_2(t) = \int_0^t \lambda_2(\tau)\mathrm{d}\tau \tag{18}$$

it readily follows from (1), (10), (15), (17), (9), (16) and (18) that

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \boldsymbol{\Phi}(t,\tau) \begin{bmatrix} x_1(\tau) \\ x_2(\tau) \end{bmatrix} \quad , \tag{19}$$

where the fundamental matrix $\boldsymbol{\Phi}$ is given by

$$\boldsymbol{\Phi}(t,\tau) = \mathbf{L}(t)\,\exp\{\mathbf{D}(t,\tau)\}\,\mathbf{L}^{-1}(\tau) \quad , \tag{20}$$

with a Lyapunov matrix

$$\mathbf{L}(t) = \begin{bmatrix} 1 & u(t) \\ l(t) & l(t)u(t)+1 \end{bmatrix} \tag{21}$$

and a diagonal matrix

$$\mathbf{D}(t,\tau) = \begin{bmatrix} \gamma_1(t)-\gamma_1(\tau) & 0 \\ 0 & \gamma_2(t)-\gamma_2(\tau) \end{bmatrix} . \tag{22}$$

3. THE SLOWLY-VARYING CONDITION

The fundamental solution of (1) under slowly-varying conditions is written in [1] and [6] as

$$\tilde{\mathbf{\Phi}}(t,\tau) \;=\; \exp\{\mathbf{A}(\tau)\,t\} \quad , \tag{23}$$

in which $\mathbf{A}$ denotes the system matrix and where $t > \tau$ with τ the 'frozen time'. However if both $t \geq 0$ and $\tau \geq 0$ independently, then (23) reads

$$\tilde{\mathbf{\Phi}}(t,\tau) \;=\; \exp\{\mathbf{A}(\tau)\,(t-\tau)\} \quad . \tag{24}$$

Of course, $\tilde{\mathbf{\Phi}}$ should be in some sense an approximation of the exact expression for $\mathbf{\Phi}(t,\tau)$ in (20). To show this is indeed the case, we write (20) as

$$\mathbf{\Phi}(t,\tau) = \mathbf{L}(t)\mathbf{L}^{-1}(\tau)\exp\{\mathbf{L}(\tau)\mathbf{D}(t,\tau)\mathbf{L}^{-1}(\tau)\}\,. \tag{25}$$

in which $\mathbf{D}(t,\tau)$ is the diagonal matrix of (22). If the slowly-varying condition is translated into the explicit approximations

$$\mathbf{L}(t) \approx \mathbf{L}(\tau) \tag{26}$$

and

$$\gamma_i(t) \approx s_i t \quad , \tag{27}$$

with s_i constants, then we conclude

$$\mathbf{\Phi}(t,\tau) \approx \tilde{\mathbf{\Phi}}(t,\tau) \quad . \tag{28}$$

Remark that the expression

$$\mathbf{A}(\tau) \approx \mathbf{L}(\tau)\frac{1}{t-\tau}\mathbf{D}(t,\tau)\mathbf{L}^{-1}(\tau) \tag{29}$$

with $\mathbf{D}(t,\tau)$ defined in (22), represents a similarity transformation which diagonalizes $\mathbf{A}(\tau)$. This similarity transformation introduces a frame of reference (with the colums of $\mathbf{L}(\tau)$ as coordinate-axes) in which the motion of the system is decoupled along the principal axes. The condition (26) states that this frame of reference does hardly move with respect to the original frame of reference.
In principle, the conditions (26) and (27) are necessary for a varying system to be approximated by an invariant system. There are, however, systems which satisfy (26) but not (27). We remark that (26) and (27) are satisfied over a time-interval I, if the coefficients of the system matrix are constant on that interval. Further, the relation (26) should hold for all columns of the system matrix, even if the system is slow in a specific direction. Consider as an illustration a system which has a circular orbit in a fixed frame of reference. Then we have

$$\mathbf{A} \;=\; \begin{bmatrix} 0 & -\dot{\phi} \\ \dot{\phi} & 0 \end{bmatrix} \tag{30}$$

with $\dot{\phi}$ an angular velocity. Note that (9) has in this case a solution

$$l(t) \;=\; \tan\phi(t) \tag{31}$$

So, a slowly-varying behaviour follows only if

$$\dot{\phi} \approx \text{constant} \tag{32}$$

It can be seen from (22) and (25) that the stability behaviour of the system is determined by the quantities $\gamma_i(t) - \gamma_i(\tau)\;(i\;=\;1,2)$. Since $\mathbf{L}(t)$ is Lyapunov, it is clear that due to boundedness of $\mathbf{L}(t)$ the system is stable if

$$\lim_{(t-\tau)\to\infty} \Re\frac{1}{t-\tau}\int_{\tau}^{t}\lambda_i(\xi)\mathrm{d}\xi \;<\; 0 \tag{33}$$

where $\Re$ denotes the real part.

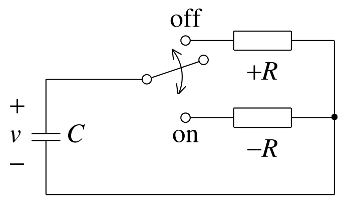

Figure 1: First order linear time-varying circuit

This is demonstrated by the circuit of figure 1. Here it is assumed that the capacitor has an initial charge and that the switch is periodically in the on/off state. Since the circuit is a first order system, the pole is alternating between left and right half-plane according to the position of the switch. If the switch is positioned equal periods in the on- and off-state, then we have

$$\lim_{(t-\tau)\to\infty} \Re\frac{1}{t-\tau}\int_{\tau}^{t}\lambda(\xi)\mathrm{d}\xi \;=\; 0 \tag{34}$$

Expressions (33) and (34) sharpen the result of Solo [6]. It states that the dynamic eigenvalues has to be an average in the left half plane in order that the system is stable. In principle, this point has nothing to do with slowly-varying.
We mention a third point for discussion. From (8) it is seen that algebraic and dynamic eigenvalues are solutions of the same characteristic equation if $\dot{\mathbf{l}} \;=\; 0$. In that case the effect of an algebraic similarity transformation which diagonalizes $\mathbf{A}$, will be derived from (1). To that aim, introduce

$$\boldsymbol{x} \;=\; \mathbf{S}\boldsymbol{y} \tag{35}$$

where $\mathbf{S}$ diagonalizes $\mathbf{A}$, $\boldsymbol{x} = [x_1\;x_2]^T$ and $\boldsymbol{y} = [y_1\;y_2]^T$, respectively. Then

$$\dot{\boldsymbol{y}} \;=\; (\mathbf{\Lambda} - \mathbf{S}^{-1}\dot{\mathbf{S}})\boldsymbol{y} \tag{36}$$

where $\mathbf{\Lambda}$ is a diagonal matrix such that

$$\mathbf{\Lambda} \;=\; \mathbf{S}^{-1}\mathbf{A}\mathbf{S} \tag{37}$$

Now it is clear that for $\dot{\mathbf{S}} = \mathbf{0}$, the similarity transformation decouples the set of equations. Iteration procedures to solve (36) with asymptotic methods for 2-dimensional systems are given [9]. The next step in the iteration procedure is to find a similarity transformation such that $\mathbf{\Lambda} - \mathbf{S}^{-1}\dot{\mathbf{S}}$ becomes diagonal. A formal proof of such an iteration procedure is given in the accompanying paper [12].

4. CONCLUSIONS

It is argued that the conditions for a system to be slowly-varying can be derived from the fundamental matrix in stead of the system matrix. Then he time-averaged dynamic eigenvalue appears to be a measure for stability of the system. These dynamic eigenvalue can be approximated by the classical algebraic eigenvalue for slowly-varying systems. This declares the results of Desoer [1] and Solo [6]. Moreover, iteration procedures for obtaining dynamic eigenvalues can be interpreted as means to make the system slower in a new frame of reference.

5. REFERENCES

[1] C.A.Desoer, "Slowly varying System $\dot{\boldsymbol{x}}=\mathbf{A}(t)\boldsymbol{x}$", *IEEE Trans. Automat. Contr.*, Vol. AC 14, pp. 780-1, Dec. 1969.

[2] C.A.Desoer, K.K.Wong, "Small-Signal Behaviour of Nonlinear Lumped Networks", *Proc. IEEE*, Vol. 56, No. 1, Jan. 1968.

[3] E.W. Kamen, P.P. Khargonekar, A. Tannenbaum, "Control of Slowly-Varying Systems", *IEEE Trans Automat. Contr.* Vol. 34, No. 12, pp. 1283-5, Dec. 1989.

[4] W.J. Rugh, *Linear System Theory*, Prentice Hall, 1993.

[5] L.Ya. Adrianova, *Introduction to Linear Systems of Differential equations*, American Math. Soc., Providence, 1995.

[6] V. Solo, "On the stability of Slowly Time-Varying Linear Systems", *Math. Control Signals Systems*, Vol. 7, pp. 331-50, 1994.

[7] P. van der Kloet, F.L. Neerhoff, F.C.M. Kuijstermans, A. van Staveren, C.J.M. Verhoeven, "Generalizations for the Eigenvalue and Pole Concept with respect to Linear Time-Varying Systems", *Proc. PRORISC/IEEE Workshop CSSP*, pp. 291-6,Mierlo, The Netherlands, Nov. 27-8, 1997.

[8] E.W.Kamen, "The poles and zeros of a linear time-varying system", *Lin. Algebra and its Appl.*, Vol. 98, pp. 263-289, 1988.

[9] M.S.P. Eastham, *The asymptotic Solution of Linear Differential Systems*, Claredon Press, Oxford, 1989.

[10] M.-Y. Wu, "A new concept of eigenvalues and its applications", *IEEE Trans. AC*, Vol. AC-25, No. 4, pp. 824-828,1980.

[11] M.-Y. Wu, "On the stability of linear time-varying systems", *Int. J. Systems Science*, Vol.15, No. 2, pp. 137-150, 1984.

[12] P. van der Kloet, F.L. Neerhoff, "Iteration schemes for the modal solutions of linear time-varying systems", *NDES' 99*, Copenhagen, Denmark, July 15-17th, 1999.

[13] P. van der Kloet, F.L. Neerhoff, "On eigenvalues and poles for second order linear time-varying systems", *Proc. NDES'97*, pp. 300-305, Moscow, 26-27 June 1997.

[14] F.C.M. Kuijstermans, A. van Staveren, P.van der Kloet, F.L. Neerhoff, C.J.M. Verhoeven, "Dynamic Exponent-based Electronics", *Proc. PRORISC/IEEE Workshop CSSP*, pp. 307-15, Mierlo, The Netherlands, Nov. 27-8, 1997.

[15] F.M. Diepstraten, F.C.M. Kuijstermans, W.A. Serdijn, P. van der Kloet, A. van Staveren, F.L. Neerhoff, C.J.M. Verhoeven and A.H.M. van Roermond, "Dynamic Behaviour of Dynamic Translinear Circuits: the Linear Time-varying Approach", submitted to *IEEE Trans. on Circuits and Systems*.

[16] P. van der Kloet, F.L. Neerhoff, "On the Generalization of the Characteristic Polynomial for Linear Time-varying Dynamic Systems", submitted to *ECCTD '99*, Stresa, Italy, Aug. 29 - Sep. 2, 1999.